Section 2.1

eg Find an expression for $\sum_{i=1}^{n} (2i-1)$ and use it to compute $\sum_{i=1}^{100} (2i-1)$.

We have already seen that

$$\sum_{i=1}^{n} (2i-1) = 2\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1$$

$$= 2 \cdot \frac{n(n+1)}{2} - n$$

$$= n^2 + n - n$$

$$\boxed{= n^2}$$

Next, $\sum_{i=1}^{100} (2i-1) = 100^2 \boxed{= 10000}$.

eg Evaluate $\sum_{i=1}^{n} (ni-4)^2$

We can write

$$\sum_{i=1}^{n} (ni-4)^2 = \sum_{i=1}^{n} (n^2 i^2 - 8ni + 16)$$

$$= n^2 \sum_{i=1}^{n} i^2 - 8n \sum_{i=1}^{n} i + 16 \sum_{i=1}^{n} 1$$

$$= n^2 \cdot \frac{n(n+1)(2n+1)}{6} - 8n \cdot \frac{n(n+1)}{2} + 16 \cdot n$$

$$\boxed{= \frac{1}{3} n^5 + \frac{1}{2} n^4 - \frac{23}{6} n^3 - 4n^2 + 16n}$$

Using sigma notation, we can rewrite the lower sum as

$$s(n) = [f(m_1) + f(m_2) + f(m_3) + \cdots + f(m_n)] \Delta x$$

$$= \left[\sum_{i=1}^{n} f(m_i)\right] \Delta x$$

$$= \sum_{i=1}^{n} f(m_i) \Delta x .$$

Likewise, the upper sum becomes

$$S(n) = [f(M_1) + f(M_2) + f(M_3) + \cdots + f(M_n)] \Delta x$$

$$= \left[\sum_{i=1}^{n} f(M_i)\right] \Delta x$$

$$= \sum_{i=1}^{n} f(M_i) \Delta x$$

eg Let's again consider the region R under $f(x) = x^2 + 1$, above the x-axis, between $x = 0$ and $x = 1$.

We will find expressions for the lower sum and the upper sum using n subintervals and use them to estimate the area A using 100 subintervals.

Assuming a regular partition, $\Delta x = \dfrac{1-0}{n} = \dfrac{1}{n}$.

Then the right endpoint of each subinterval is

$$x_i = 0 + i\Delta x = \frac{i}{n}$$

and thus the left endpoint is $x_{i-1} = \dfrac{i-1}{n}$.

Since $f(x)$ is increasing the minimum value on each subinterval must occur at the left endpoint x_{i-1} so

$$M_i = \frac{i-1}{n}$$

$$f(M_i) = \left(\frac{i-1}{n}\right)^2 + 1 = \frac{i^2}{n^2} - \frac{2i}{n^2} + \frac{1}{n^2} + 1$$

Thus the lower sum is given by

$$s(n) = \sum_{i=1}^{n} \left(\frac{i^2}{n^2} - \frac{2i}{n^2} + \frac{1}{n^2} + 1\right) \cdot \frac{1}{n}$$

$$= \sum_{i=1}^{n} \left(\frac{i^2}{n^3} - \frac{2i}{n^3} + \frac{1}{n^3} + \frac{1}{n}\right)$$

$$= \frac{1}{n^3} \sum_{i=1}^{n} i^2 - \frac{2}{n^3} \sum_{i=1}^{n} i + \frac{1}{n^3} \sum_{i=1}^{n} 1 + \frac{1}{n} \sum_{i=1}^{n} 1$$

$$= \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{2}{n^3} \cdot \frac{n(n+1)}{2} + \frac{1}{n^3} \cdot n + \frac{1}{n} \cdot n$$

$$= \frac{4}{3} - \frac{1}{2n} + \frac{1}{6n^2}$$

$s(100) \approx 1.328$ Compare: $s(4) = 1.22$

The maximum value on each subinterval occurs at the right endpoint x_i so

$$M_i = \frac{i}{n}$$

$$f(M_i) = \left(\frac{i}{n}\right)^2 + 1 = \frac{i^2}{n^2} + 1$$

The upper sum will be

$$S'(n) = \sum_{i=1}^{n} \left(\frac{i^2}{n^2} + 1\right) \cdot \frac{1}{n}$$

$$= \sum_{i=1}^{n} \left(\frac{i^2}{n^3} + \frac{1}{n}\right)$$

$$= \frac{1}{n^3} \sum_{i=1}^{n} i^2 + \frac{1}{n} \sum_{i=1}^{n} 1$$

$$= \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{1}{n} \cdot n$$

$$= \frac{4}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$

$S'(100) \approx 1.338$ Compare: $S'(4) = 1.47$

In fact, observe that

$$\lim_{n\to\infty} s(n) = \lim_{n\to\infty} \left(\frac{4}{3} - \frac{1}{2n} + \frac{1}{6n^2}\right) = \frac{4}{3} - 0 + 0$$
$$= \frac{4}{3}$$

$$\lim_{n\to\infty} S'(n) = \lim_{n\to\infty} \left(\frac{4}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right) = \frac{4}{3} + 0 + 0$$
$$= \frac{4}{3}$$

Thus we can conclude that

$$A = \lim_{n\to\infty} s(n) = \lim_{n\to\infty} S'(n) \boxed{= \frac{4}{3}}$$

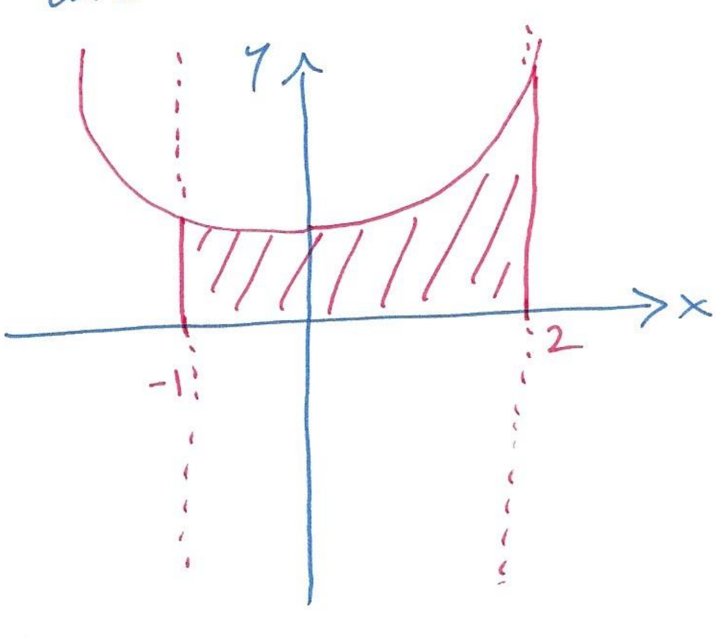
To compute A, we can choose any <u>sample point</u> x_i^* which lies on the appropriate subinterval and set the height of each rectangle to be $f(x_i^*)$. Since we have freedom in terms of how we choose x_i^*, we typically want it to have the simplest possible form, so we normally choose $x_i^* = x_i$ (the right endpoint).

In general, then we have found that

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_i^* = a + i\Delta x$.

eg Find the area under $f(x) = x^2+1$, above the x-axis, and between $x=-1$ and $x=2$.



We will choose

$$\Delta x = \frac{2-(-1)}{n} = \frac{3}{n}$$

$$x_i^* = x_i = -1 + \frac{3i}{n}$$

$$f(x_i^*) = \left(-1+\frac{3i}{n}\right)^2 + 1$$

$$= 1 - \frac{6i}{n} + \frac{9i^2}{n^2} + 1$$

$$= \frac{9i^2}{n^2} - \frac{6i}{n} + 2$$