Section 2.1

Suppose we want to approximate A using n rectangles. Then we partition the interval $[a, b]$ into n subintervals of width

$$\Delta x = \frac{b-a}{n}.$$

Then the first subinterval is $[a, a+\Delta x]$

second subinterval is $[a+\Delta x, a+2\Delta x]$

$\vdots$

ith subinterval is $[a+(i-1)\Delta x, a+i\Delta x]$.

$\vdots$

nth subinterval is $[a+(n-1)\Delta x, a+n\Delta x]$

where $$a+n\Delta x = a+n\cdot\frac{b-a}{n} = a+(b-a) = b.$$

We call the right endpoint of the ith subinterval x_i

where $$x_i = a + i\Delta x$$

Note that the left endpoint of the ith subinterval is, therefore, x_{i-1}.

We can then denote $x_0 = a$ and $x_n = b$.

Now we assume that $f(x)$ is continuous on $[a,b]$ and therefore on each subinterval. Then the Extreme Value Theorem guarantees that $f(x)$ will have both a minimum value and a maximum value on each subinterval.

Let $x = m_i$ be the absolute minimum of $f(x)$ on the ith subinterval, so $f(m_i)$ is the minimum value. Then we would choose the height of each rectangle to be $f(m_i)$ in order to compute the lower sum.

Similarly let $x = M_i$ be the absolute maximum of $f(x)$ on the ith subinterval, so $f(M_i)$ is the maximum value. Then we choose $f(M_i)$ to be the height of the rectangles to obtain the upper sum.

In general, then, the lower sum is given by

$$s(n) = f(m_1)\Delta x + f(m_2)\Delta x + f(m_3)\Delta x + \cdots + f(m_n)\Delta x$$

$$= \left[f(m_1) + f(m_2) + f(m_3) + \cdots + f(m_n)\right]\Delta x$$

The upper sum is given by

$$S(n) = f(M_1)\Delta x + f(M_2)\Delta x + f(M_3)\Delta x + \cdots + f(M_n)\Delta x$$

$$= \left[f(M_1) + f(M_2) + f(M_3) + \cdots + f(M_n)\right]\Delta x$$

Consider a sum of n terms $a_1, a_2, a_3, \ldots, a_n$ for which all of the terms can be described by a general formula a_i. Here we call i the index of summation. Then we can write the sum in sigma notation as follows:

$$a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

Here, we call a_i the summand, $i=1$ the lower bound of summation, and n the upper bound of summation.

eg $1 + 4 + 9 + 16 + \cdots + 144$

All of the terms can be written in the form i^2. Thus $a_i = i^2$ with lower bound 1 and upper bound 12. Hence we can write this sum in sigma notation as $\sum_{i=1}^{12} i^2$.

eg $$2 + \frac{3}{8} + \frac{4}{27} + \frac{5}{64} + \cdots + \frac{n+1}{n^3} = \sum_{i=1}^{n} \frac{i+1}{i^3}$$

Theorem: Basic Properties of Sums

① $\sum_{i=1}^{n} k a_i = k \sum_{i=1}^{n} a_i$ for any constant k

② $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

③ $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

However, $\sum_{i=1}^{n} a_i b_i \neq \left(\sum_{i=1}^{n} a_i\right) \cdot \left(\sum_{i=1}^{n} b_i\right)$

$$\sum_{i=1}^{n} \frac{a_i}{b_i} \neq \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}$$

eg $\sum_{i=1}^{n} (2i - 1) = \sum_{i=1}^{n} 2i - \sum_{i=1}^{n} 1$

$$= 2 \sum_{i=1}^{n} i - \sum_{i=1}^{n} 1$$

Theorem: Summation Formulas

① $\sum_{i=1}^{n} 1 = \underbrace{1 + 1 + 1 + 1 + \cdots + 1}_{n \text{ times}} = n$

② $\sum_{i=1}^{n} i = 1 + 2 + 3 + 4 + \cdots + n = \frac{n(n+1)}{2}$

③ $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

④ $\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$