Section 1.4

eg $\int x^3 \cosh(x^2)\,dx$

Let $u = x^2$ so $du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

The integral becomes

$$\int x^3 \cosh(x^2)\,dx = \int x^2 \cosh(x^2)\cdot x\,dx$$
$$= \int u \cosh(u)\cdot \frac{1}{2}du$$
$$= \frac{1}{2}\int u \cosh(u)\,du$$

Now we apply integration by parts with

$w = u$ $\qquad dw = 1\cdot du = du$

$dv = \cosh(u)\,du$ $\qquad v = \sinh(u)$

So we have

$$\int x^3 \cosh(x^2)\,dx = \frac{1}{2}\left[u\sinh(u) - \int \sinh(u)\,du\right]$$
$$= \frac{1}{2}u\sinh(u) - \frac{1}{2}\int \sinh(u)\,du$$
$$= \frac{1}{2}u\sinh(u) - \frac{1}{2}\cosh(u) + C$$
$$\boxed{= \frac{1}{2}x^2\sinh(x^2) - \frac{1}{2}\cosh(x^2) + C}$$

Guidelines for indefinite integration (a work in progress...)

Given an integral $\int f(x)dx$, we consider the following in deciding how to evaluate the integral:

① Is this an elementary integral? This includes common integrals, inverse trigonometric integrals, and any combination of them for which the Basic Properties apply, as well as instances of linear composition.

② Can it be rewritten in terms of elementary integrals? We could do this algebraically, using trigonometric identities, using long division, by completing the square, etc.

③ Is u-substitution appropriate?

④ Is integration by parts appropriate?

⑤ Does the integral require a combination of techniques?

eg $\int \sin(\sqrt{x+2})\,dx$

Let $u = \sqrt{x+2}$

$$du = \tfrac{1}{2}(x+2)^{-1/2}\,dx = \frac{1}{2\sqrt{x+2}}\,dx$$

$$= \frac{1}{2u}\,dx \rightarrow dx = 2u\,du$$

The integral becomes

$$\int \sin(\sqrt{x+2})\,dx = \int \sin(u)\cdot 2u\,du$$
$$= 2\int u\sin(u)\,du$$

Now we use integration by parts with

$$w = u \qquad dw = 1\cdot du = du$$
$$dv = \sin(u)\,du \qquad v = -\cos(u)$$

so then

$$\int \sin(\sqrt{x+2})\,dx = 2\left[-u\cos(u) + \int \cos(u)\,du\right]$$
$$= 2\left[-u\cos(u) + \sin(u)\right] + C$$
$$\boxed{= -2\sqrt{x+2}\cos(\sqrt{x+2}) + 2\sin(\sqrt{x+2}) + C}$$

In some cases we can use integration by parts to derive a reduction formula, which gives the result of applying this method to integrals of a certain form any number of times.

eg Derive the reduction formula

$$\int x^n e^x\,dx = x^n e^x - n\int x^{n-1} e^x\,dx$$

where n is any natural number. Use it to evaluate $\int x^3 e^x\,dx$.

We use integration by parts with

$$w = x^n \qquad dw = nx^{n-1}\,dx$$

$$dv = e^x\,dx \qquad v = e^x$$

so

$$\int x^n e^x\,dx = x^n e^x - \int e^x \cdot n x^{n-1}\,dx$$

$$= x^n e^x - n\int x^{n-1} e^x\,dx.$$

Next,

$$\int x^3 e^x\,dx = x^3 e^x - 3\int x^2 e^x\,dx$$

$$= x^3 e^x - 3\left[x^2 e^x - 2\int x e^x\,dx\right]$$

$$= x^3 e^x - 3x^2 e^x + 6\int x e^x\,dx$$

$$= x^3 e^x - 3x^2 e^x + 6\left[x e^x - \int e^x\,dx\right]$$

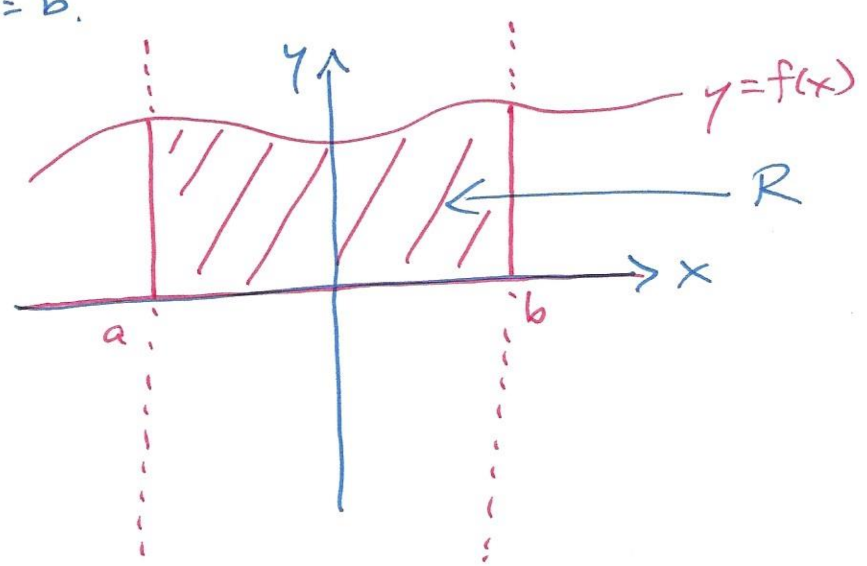
$$= x^3 e^x - 3x^2 e^x + 6x e^x - 6\int e^x\,dx$$

$$\boxed{= x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C}$$

Section 2.1: Area Under a Curve

We know how to find the area of elementary shapes like rectangles, triangles and circles. But how can we find the area of a region that is <u>not</u> simply a combination of these shapes?

Specifically, consider a region R which is bounded above by a curve $y = f(x)$, bounded below by the x-axis, bounded to the left by $x = a$, and bounded to the right by $x = b$.



We want to determine the exact value of the area A of the region R.