MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Differential Equations **Math 1001 Worksheet** Fall 2019

SOLUTIONS

1. (a) We rewrite and integrate to obtain the general solution:

$$f'(x) = \cos(2x-1) + 2$$
$$\int f'(x)\,dx = \int [\cos(2x-1)+2]\,dx$$
$$f(x) = \frac{1}{2}\sin(2x-1) + 2x + C.$$

To find the particular solution, we first observe that

$$f\left(\frac{1}{2}\right) = \frac{1}{2}\sin\left(2\left(\frac{1}{2}\right)-1\right) + 2\left(\frac{1}{2}\right) + C = \frac{1}{2}\sin(0) + 1 + C = 1 + C.$$

But we are given that $f\left(\frac{1}{2}\right) = 3$, so we must have $1 + C = 3$ and hence $C = 2$. Thus the particular solution is

$$f(x) = \frac{1}{2}\sin(2x-1) + 2x + 2.$$

[6] (b) Again, we rewrite and integrate:

$$f''(x) = 4x^{-2}$$
$$\int f''(x)\,dx = 4\int x^{-2}\,dx$$
$$f'(x) = 4\left[\frac{1}{-1}x^{-1}\right] + C = -\frac{4}{x} + C.$$

This gives $f'(1) = -4 + C = 0$ and so $C = 4$. Now we integrate a second time:

$$\int f'(x)\,dx = \int (-4x^{-1} + 4)\,dx$$
$$f(x) = -4\ln|x| + 4x + C$$

yielding $f(-1) = -4\ln|-1| + 4(-1) + C = -4 + C$. Then we can set $-4 + C = 3$ to get $C = 7$, and the particular solution is

$$f(x) = -4\ln|x| + 4x + 7.$$

(c) Integrating twice gives

$$\begin{aligned}\int f''(x)\,dx &= \int (3x-3)\,dx\\ f'(x) &= \frac{3}{2}x^2 - 3x + C\\ \int f'(x)\,dx &= \int \left(\frac{3}{2}x^2 - 3x + C\right)\,dx\\ f(x) &= \frac{3}{2}\left[\frac{1}{3}x^3\right] - 3\left[\frac{1}{2}x^2\right] + Cx + D = \frac{1}{2}x^3 - \frac{3}{2}x^2 + Cx + D,\end{aligned}$$

where both C and D are arbitrary constants. Using the first initial condition, we have that $f(0) = D = -5$. Using the other condition, we get $f(2) = \frac{1}{2}(8) - \frac{3}{2}(4) + C(2) - 5 = 4 - 6 + 2C - 5 = 2C - 7$. Then we set $2C - 7 = -7$ to get $C = 0$. Hence the particular solution is

$$\frac{1}{2}x^3 - \frac{3}{2}x^2 - 5.$$

2. Integrating gives

$$\begin{aligned}\int f'(x)\,dx &= \int 9x^2\,dx\\ f(x) &= 9\left[\frac{1}{3}x^3\right] + C = 3x^3 + C.\end{aligned}$$

We want the line $y = 36x$ to be tangent to the graph $y = f(x)$, that is, to $y = 3x^3 + C$. This means that the two curves must meet at a point where their slopes are equal. But the slope of $y = 36x$ is always $y' = 36$, so we solve $f'(x) = 36$, giving

$$9x^2 = 36 \quad\Longrightarrow\quad x^2 = 4 \quad\Longrightarrow\quad x = \pm 2.$$

In the first case, from the equation of the line we have $y = 36(2) = 72$ so then

$$3(2)^3 + C = 72 \quad\Longrightarrow\quad 24 + C = 72 \quad\Longrightarrow\quad C = 48.$$

In the second case, we have $y = 36(-2) = -72$ and thus

$$3(-2)^3 + C = -72 \quad\Longrightarrow\quad -24 + C = -72 \quad\Longrightarrow\quad C = -48.$$

Hence the two such functions are

$$f(x) = 3x^3 + 48 \quad\text{and}\quad f(x) = 3x^3 - 48.$$

3. (a) The acceleration function is simply $a(t) = -9.8$, so integrating twice gives us both the

velocity and position functions:

$$\begin{aligned}
\int a(t)\,dt &= \int (-9.8)\,dt \\
v(t) &= -9.8t + C \\
\int v(t)\,dt &= \int (-9.8t + C)\,dt \\
s(t) &= -9.8\left[\frac{1}{2}t^2\right] + Ct + D = -4.9t^2 + Ct + D.
\end{aligned}$$

We are told that the rocket is launched from the ground, which implies that $s(0) = 0$, and so $D = 0$. Now let the time at which the rocket reaches its maximum height be T; then $v(T) = 0$ and we have that $-9.8T + C = 0$ so $T = \frac{C}{9.8}$. We want $s(T) = 4410$, so then

$$\begin{aligned}
s(T) &= -4.9T^2 + CT \\
4410 &= -4.9\left(\frac{C}{9.8}\right)^2 + C\left(\frac{C}{9.8}\right) \\
4410 &= \frac{C^2}{19.6} \\
C^2 &= 86436 \\
C &= \pm 294.
\end{aligned}$$

Finally, we have the initial velocity $v(0) = C = \pm 294$. Since the rocket is launched upward, we can accept only the positive answer; hence the initial velocity must be 294 metres per second.

(b) From the above, the rocket reaches its maximum height when $T = \frac{C}{9.8} = \frac{294}{9.8} = 30$, that is, after 30 seconds.

(c) The particular solution is $s(t) = -4.9t^2 + 294t$, so

$$s(10) = -4.9(100) + 294(10) = 2450.$$

The rocket is 2450 metres high after 10 seconds.