MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 8 **MATHEMATICS 1001** FALL 2019

SOLUTIONS

[5] 1. (a) We let $x = 3\tan(\theta)$ so $dx = 3\sec^2(\theta)\,d\theta$. Then

$$\sqrt{x^2+9} = \sqrt{9\tan^2(\theta)+9} = \sqrt{9\sec^2(\theta)} = 3\sec(\theta)$$

and the integral becomes

$$\begin{aligned}
\int \frac{\sqrt{x^2+9}}{x}\,dx &= \int \frac{3\sec(\theta)}{3\tan(\theta)}\cdot 3\sec^2(\theta)\,d\theta \\
&= 3\int \frac{\sec^3(\theta)}{\tan(\theta)}\,d\theta \\
&= 3\int \frac{[1+\tan^2(\theta)]\sec(\theta)}{\tan(\theta)}\,d\theta \\
&= 3\int \left[\frac{\sec(\theta)}{\tan(\theta)} + \tan(\theta)\sec(\theta)\right]d\theta \\
&= 3\int \left[\csc(\theta) + \frac{\sin(\theta)}{\cos^2(\theta)}\right]d\theta \\
&= -3\ln|\csc(\theta)+\cot(\theta)| + 3\int \frac{\sin(\theta)}{\cos^2(\theta)}\,d\theta.
\end{aligned}$$

For the remaining integral, let $u = \cos(\theta)$ so $-\,du = \sin(\theta)\,d\theta$. Then

$$\int \frac{\sin(\theta)}{\cos^2(\theta)}\,d\theta = -\int \frac{1}{u^2}\,du = \frac{1}{u} + C = \frac{1}{\cos(\theta)} + C = \sec(\theta) + C$$

and so

$$\int \frac{\sqrt{x^2+9}}{x}\,dx = -3\ln|\csc(\theta)+\cot(\theta)| + 3\sec(\theta) + C.$$

From the substitutions, we immediately have

$$\tan(\theta) = \frac{x}{3} \quad\text{and}\quad \sec(\theta) = \frac{1}{3}\sqrt{x^2+9}.$$

Thus

$$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{3}{x} \quad\text{and}\quad \csc(\theta) = \frac{\sec(\theta)}{\tan(\theta)} = \frac{\sqrt{x^2+9}}{x}.$$

Hence

$$\begin{aligned}
\int \frac{\sqrt{x^2+9}}{x}\,dx &= -3\ln\left|\frac{\sqrt{x^2+9}}{x} + \frac{3}{x}\right| + \sqrt{x^2+9} + C \\
&= -3\ln\left|\frac{\sqrt{x^2+9}+3}{x}\right| + \sqrt{x^2+9} + C.
\end{aligned}$$

[5] (b) First we complete the square:

$$x^2 - 6x - 7 = (x^2 - 6x + 9) - 7 - 9 = (x-3)^2 - 16.$$

Thus

$$\int \frac{x}{\sqrt{x^2-6x-7}}\,dx = \int \frac{x}{\sqrt{(x-3)^2-16}}\,dx.$$

Now we let $x - 3 = 4\sec(\theta)$ so $x = 4\sec(\theta) + 3$, $dx = 4\sec(\theta)\tan(\theta)\,d\theta$ and

$$\sqrt{(x-3)^2-16} = \sqrt{16\sec^2(\theta)-16} = \sqrt{16\tan^2(\theta)} = 4\tan(\theta).$$

Now the integral becomes

$$\begin{aligned}\int \frac{x}{\sqrt{x^2-6x-7}}\,dx &= \int \frac{4\sec(\theta)+3}{4\tan(\theta)} \cdot 4\sec(\theta)\tan(\theta)\,d\theta \\ &= \int [4\sec^2(\theta) + 3\sec(\theta)]\,d\theta \\ &= 4\tan(\theta) + 3\ln|\sec(\theta)+\tan(\theta)| + C \\ &= \sqrt{x^2-6x-7} + 3\ln\left|\frac{1}{4}x - \frac{3}{4} + \frac{1}{4}\sqrt{x^2-6x-7}\right| + C.\end{aligned}$$

Note that we could clean this up a bit by using the properties of logarithms and writing

$$\begin{aligned}\int \frac{x}{\sqrt{x^2-6x-7}}\,dx &= \sqrt{x^2-6x-7} + 3\ln\left|\frac{1}{4}\left(x-3+\sqrt{x^2-6x-7}\right)\right| + C \\ &= \sqrt{x^2-6x-7} + 3\ln\left|\frac{1}{4}\right| + 3\ln\left|\left(x-3+\sqrt{x^2-6x-7}\right)\right| + C \\ &= \sqrt{x^2-6x-7} + 3\ln\left|\left(x-3+\sqrt{x^2-6x-7}\right)\right| + C,\end{aligned}$$

where we have absorbed the constant term $3\ln\left|\frac{1}{4}\right|$ into the arbitrary constant.

[5] (c) Since we can write

$$\int x\sqrt{25-x^4}\,dx = \int x\sqrt{25-(x^2)^2}\,dx,$$

let $x^2 = 5\sin(\theta)$ so $2x\,dx = 5\cos(\theta)\,d\theta$ and $x\,dx = \frac{5}{2}\cos(\theta)\,d\theta$. Thus

$$\sqrt{25-x^4} = \sqrt{25-25\sin^2(\theta)} = 5\cos(\theta)$$

and the integral becomes

$$\begin{aligned}\int x\sqrt{25-x^4}\,dx &= \int 5\cos(\theta)\cdot\frac{5}{2}\cos(\theta)\,d\theta \\ &= \frac{25}{2}\int\cos^2(\theta)\,d\theta \\ &= \frac{25}{2}\int\frac{1+\cos(2\theta)}{2}\,d\theta \\ &= \frac{25}{4}\left[\theta+\frac{1}{2}\sin(2\theta)\right]+C \\ &= \frac{25}{4}\theta+\frac{25}{4}\sin(\theta)\cos(\theta)+C \\ &= \frac{25}{4}\arcsin\left(\frac{x^2}{5}\right)+\frac{25}{4}\cdot\frac{x^2}{5}\cdot\frac{1}{5}\sqrt{25-x^4}+C \\ &= \frac{25}{4}\arcsin\left(\frac{x^2}{5}\right)+\frac{1}{4}x^2\sqrt{25-x^4}+C.\end{aligned}$$

[5] (d) Let $x=\sin(\theta)$ so $dx=\cos(\theta)\,d\theta$ and

$$\sqrt{1-x^2}=\sqrt{1-\sin^2(\theta)}=\sqrt{\cos^2(\theta)}=\cos(\theta).$$

When $x=\frac{\sqrt{2}}{2}$, we have

$$\theta=\arcsin\left(\frac{\sqrt{2}}{2}\right)=\frac{\pi}{4}.$$

When $x=\frac{\sqrt{3}}{2}$, we have

$$\theta=\arcsin\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{3}.$$

The integral becomes

$$\begin{aligned}\int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}}\frac{x^3}{\sqrt{1-x^2}}\,dx &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sin^3(\theta)}{\cos(\theta)}\cdot\cos(\theta)\,d\theta \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\sin^3(\theta)\,d\theta \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}}[1-\cos^2(\theta)]\sin(\theta)\,d\theta.\end{aligned}$$

Let $u = \cos(\theta)$ so $-du = \sin(\theta)\, d\theta$. When $\theta = \frac{\pi}{4}$, $u = \frac{\sqrt{2}}{2}$. When $\theta = \frac{\pi}{3}$, $u = \frac{1}{2}$. Thus

$$\begin{aligned}\int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \frac{x^3}{\sqrt{1-x^2}}\, dx &= -\int_{\frac{\sqrt{2}}{2}}^{\frac{1}{2}} [1-u^2]\, du \\ &= -\left[u - \frac{1}{3}u^3\right]_{\frac{\sqrt{2}}{2}}^{\frac{1}{2}} \\ &= \frac{5\sqrt{2}}{12} - \frac{11}{24}.\end{aligned}$$

[5] 2. (a) We use integration by parts with $w = x$ so $dw = dx$, and $dv = \cos^2(x)\, dx$. Since

$$\int \cos^2(x)\, dx = \int \frac{1+\cos(2x)}{2}\, dx = \frac{1}{2}\left[x + \frac{1}{2}\sin(2x)\right] + C,$$

we have that $v = \frac{1}{2}x + \frac{1}{4}\sin(2x)$. Now

$$\begin{aligned}\int x\cos^2(x)\, dx &= x\left[\frac{1}{2}x + \frac{1}{4}\sin(2x)\right] - \int \left[\frac{1}{2}x + \frac{1}{4}\sin(2x)\right] dx \\ &= \frac{1}{2}x^2 + \frac{1}{4}x\sin(2x) - \frac{1}{2}\left(\frac{1}{2}x^2\right) - \frac{1}{4}\left[-\frac{1}{2}\cos(2x)\right] + C \\ &= \frac{1}{4}x^2 + \frac{1}{4}x\sin(2x) + \frac{1}{8}\cos(2x) + C.\end{aligned}$$

[5] (b) Since this is a product of sine and cosine functions with an odd power of sine, we write

$$\begin{aligned}\int \sin^3(x)\cos^2(x)\, dx &= \int \sin^2(x)\cos^2(x)\cdot\sin(x)\, dx \\ &= \int [1-\cos^2(x)]\cos^2(x)\cdot\sin(x)\, dx.\end{aligned}$$

Now we let $u = \cos(x)$ so $-du = \sin(x)\, dx$. The integral becomes

$$\begin{aligned}\int \sin^3(x)\cos^2(x)\, dx &= -\int [1-u^2]u^2\, du \\ &= -\int (u^2 - u^4)\, du \\ &= -\left(\frac{1}{3}u^3 - \frac{1}{5}u^5\right) + C \\ &= \frac{1}{5}\cos^5(x) - \frac{1}{3}\cos^3(x) + C.\end{aligned}$$

[5] (c) We let $u = \cos(x)$ so $-du = \sin(x)\, dx$. The integral becomes

$$\int \frac{\sin(x)}{\cos^3(x) + \cos^2(x)}\, dx = -\int \frac{1}{u^3 + u^2}\, du = -\int \frac{1}{u^2(u+1)}\, du.$$

Now we can decompose the integrand into partial fractions:

$$\frac{1}{u^2(u+1)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u+1}$$

so

$$1 = Au(u+1) + B(u+1) + Cu^2.$$

When $u = 0$, we have $1 = B$. When $u = -1$, we have $1 = C$. When, say, $u = 1$ we have $1 = 2A + 2B + C$ so $1 = 2A + 3$ and $A = -1$. Thus

$$\begin{aligned}\int \frac{\sin(x)}{\cos^3(x) + \cos^2(x)}\,dx &= -\int \left(-\frac{1}{u} + \frac{1}{u^2} + \frac{1}{u+1}\right) du \\ &= -\left[-\ln|u| - \frac{1}{u} + \ln|u+1|\right] + C \\ &= \ln|\cos(x)| + \frac{1}{\cos(x)} - \ln|\cos(x) + 1| + C \\ &= \ln|\cos(x)| + \sec(x) - \ln|\cos(x) + 1| + C.\end{aligned}$$

[5] 3. (a) First we write

$$\int_1^\infty \frac{\ln(x)}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{\ln(x)}{x^2}\,dx.$$

Now we use integration by parts with $w = \ln(x)$ so $dw = \frac{1}{x}\,dx$, and $dv = \frac{1}{x^2}\,dx$ so $v = -\frac{1}{x}$. Then

$$\begin{aligned}\int_1^\infty \frac{\ln(x)}{x^2}\,dx &= \lim_{t\to\infty} \left(\left[-\frac{\ln(x)}{x}\right]_1^t + \int_1^t \frac{1}{x^2}\,dx\right) \\ &= \lim_{t\to\infty} \left[-\frac{\ln(x)}{x} - \frac{1}{x}\right]_1^t \\ &= \lim_{t\to\infty} \left[-\frac{\ln(t)}{t} - \frac{1}{t} + 0 + 1\right] \\ &= 1 - \lim_{t\to\infty} \frac{\ln(t)}{t} - \lim_{t\to\infty} \frac{1}{t} \\ &= 1 - \lim_{t\to\infty} \frac{\ln(t)}{t} - 0.\end{aligned}$$

For the remaining limit, we use l'Hôpital's Rule:

$$\lim_{t\to\infty} \frac{\ln(t)}{t} \overset{\text{H}}{=} \lim_{t\to\infty} \frac{\frac{1}{t}}{1} = 0.$$

Thus

$$\int_1^\infty \frac{\ln(x)}{x^2}\,dx = 1 - 0 = 1.$$

[5] (b) Since the integrand is discontinuous at the left endpoint $x = 1$, We write

$$\int_1^e \frac{1}{x\ln(x)}\,dx = \lim_{t\to 1^+}\int_t^e \frac{1}{x\ln(x)}\,dx.$$

Let $u = \ln(x)$ so $du = \frac{1}{x}\,dx$. When $x = t$, $u = \ln(t)$. When $x = e$, $u = \ln(e) = 1$. So the integral becomes

$$\begin{aligned}\int_1^e \frac{1}{x\ln(x)}\,dx &= \lim_{t\to 1^+}\int_{\ln(t)}^1 \frac{1}{u}\,du\\ &= \lim_{t\to 1^+}\Big[\ln|u|\Big]_{\ln(t)}^1\\ &= \lim_{t\to 1^+}[\ln(1) - \ln|\ln(t)|]\\ &= -\lim_{t\to 1^+}\ln|\ln(t)|.\end{aligned}$$

Now, as $t \to 1^+$, $\ln(t) \to 0$, and so $\ln|\ln(t)| \to -\infty$. Hence this integral is divergent.

[5] (c) The integrand is discontinuous at $x = 5$, so first we have to rewrite this as the sum of two integrals:

$$\int_{-3}^6 \frac{1}{\sqrt[3]{x-5}}\,dx = \int_{-3}^5 \frac{1}{\sqrt[3]{x-5}}\,dx + \int_5^6 \frac{1}{\sqrt[3]{x-5}}\,dx.$$

For the first of these integrals, we write

$$\begin{aligned}\int_{-3}^5 \frac{1}{\sqrt[3]{x-5}}\,dx &= \lim_{t\to 5^-}\int_{-3}^t \frac{1}{\sqrt[3]{x-5}}\,dx\\ &= \lim_{t\to 5^-}\frac{3}{2}\Big[(x-5)^{\frac{2}{3}}\Big]_{-3}^t\\ &= \frac{3}{2}\lim_{t\to 5^-}[(t-5)^{\frac{2}{3}} - (-8)^{\frac{2}{3}}]\\ &= \frac{3}{2}[0-4]\\ &= -6.\end{aligned}$$

Similarly, for the second integral,

$$\begin{aligned}\int_5^6 \frac{1}{\sqrt[3]{x-5}}\,dx &= \lim_{t\to 5^+}\int_t^6 \frac{1}{\sqrt[3]{x-5}}\,dx\\ &= \lim_{t\to 5^+}\frac{3}{2}\Big[(x-5)^{\frac{2}{3}}\Big]_t^6\\ &= \frac{3}{2}\lim_{t\to 5^+}[1 - (t-5)^{\frac{2}{3}}]\\ &= \frac{3}{2}[1-0]\\ &= \frac{3}{2}.\end{aligned}$$

Therefore

$$\int_{-3}^{6} \frac{1}{\sqrt[3]{x-5}}\,dx = -6 + \frac{3}{2} = -\frac{9}{2}.$$