

# MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

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IMPROPER INTEGRALS

Math 1001 Worksheet

FALL 2019

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## SOLUTIONS

1. (a) We have

$$\begin{aligned}\int_2^{\infty} \frac{1}{\sqrt{4x+1}} dx &= \lim_{t \rightarrow \infty} \int_2^t \frac{1}{\sqrt{4x+1}} dx = \lim_{t \rightarrow \infty} \left[ \frac{1}{2} \sqrt{4x+1} \right]_2^t \\ &= \lim_{t \rightarrow \infty} \left( \frac{1}{2} \sqrt{4t+1} - \frac{3}{2} \right) = \infty.\end{aligned}$$

So the integral diverges.

(b) We write

$$\int_0^4 \frac{1}{\sqrt{4-x}} dx = \lim_{t \rightarrow 4^-} \int_0^t \frac{1}{\sqrt{4-x}} dx = \lim_{t \rightarrow 4^-} [-2\sqrt{4-x}]_0^t = \lim_{t \rightarrow 4^-} (-2\sqrt{4-t} + 4) = 4.$$

So the integral converges.

(c) We have

$$\int_{-\infty}^0 e^{3x} dx = \lim_{t \rightarrow -\infty} \int_t^0 e^{3x} dx = \lim_{t \rightarrow -\infty} \left[ \frac{1}{3} e^{3x} \right]_t^0 = \lim_{t \rightarrow -\infty} \left( \frac{1}{3} - \frac{1}{3} e^{3t} \right) = \frac{1}{3} - 0 = \frac{1}{3}.$$

So the integral converges.

(d) We write

$$\begin{aligned}\int_1^{\infty} \frac{1}{(x+3)^{\frac{3}{2}}} dx &= \lim_{t \rightarrow \infty} \int_1^t (x+3)^{-\frac{3}{2}} dx = \lim_{t \rightarrow \infty} \left[ \frac{-2}{\sqrt{x+3}} \right]_1^t = \lim_{t \rightarrow \infty} \left( \frac{-2}{\sqrt{t+3}} + 1 \right) \\ &= -0 + 1 = 1.\end{aligned}$$

So the integral converges.

(e) We have

$$\begin{aligned}\int_0^{\infty} \frac{x}{x^4+1} dx &= \lim_{t \rightarrow \infty} \int_0^t \frac{x}{1+(x^2)^2} dx = \lim_{t \rightarrow \infty} \left[ \frac{1}{2} \arctan(x^2) \right]_0^t \\ &= \lim_{t \rightarrow \infty} \left( \frac{1}{2} \arctan(t^2) - 0 \right) = \frac{1}{2} \left( \frac{\pi}{2} \right) = \frac{\pi}{4}.\end{aligned}$$

So the integral converges.

(f) We write

$$\begin{aligned}\int_{-\infty}^0 \frac{e^x}{1+e^x} dx &= \lim_{t \rightarrow -\infty} \int_t^0 \frac{e^x}{1+e^x} dx = \lim_{t \rightarrow -\infty} [\ln(1+e^x)]_t^0 = \lim_{t \rightarrow -\infty} [\ln(2) - \ln(1+e^t)] \\ &= \ln(2) - \ln(1+0) = \ln(2).\end{aligned}$$

So the integral converges.

(g) We have

$$\begin{aligned}\int_0^3 \frac{1}{\sqrt{9-x^2}} dx &= \lim_{t \rightarrow 3^-} \int_0^t \frac{1}{\sqrt{9-x^2}} dx = \lim_{t \rightarrow 3^-} \left[ \arcsin\left(\frac{x}{3}\right) \right]_0^t \\ &= \lim_{t \rightarrow 3^-} \left( \arcsin\left(\frac{t}{3}\right) - \arcsin(0) \right) = \frac{\pi}{2} - 0 = \frac{\pi}{2}.\end{aligned}$$

So the integral converges.

(h) We write

$$\begin{aligned}\int_0^3 \frac{x}{\sqrt{9-x^2}} dx &= \lim_{t \rightarrow 3^-} \int_0^t \frac{x}{\sqrt{9-x^2}} dx = \lim_{t \rightarrow 3^-} \left[ -\sqrt{9-x^2} \right]_0^t = \lim_{t \rightarrow 3^-} \left( -\sqrt{9-t^2} + 3 \right) \\ &= 0 + 3 = 3.\end{aligned}$$

So the integral converges.

(i) We have

$$\begin{aligned}\int_e^\infty \frac{1}{x \ln^2(x)} dx &= \lim_{t \rightarrow \infty} \int_e^t \frac{1}{x \ln^2(x)} dx = \lim_{t \rightarrow \infty} \left[ -\frac{1}{\ln(x)} \right]_e^t = \lim_{t \rightarrow \infty} \left( -\frac{1}{\ln(t)} + 1 \right) \\ &= -0 + 1 = 1.\end{aligned}$$

So the integral converges.

(j) We write

$$\begin{aligned}\int_{-\infty}^{\frac{3}{2}} \frac{1}{4x^2+9} dx &= \lim_{t \rightarrow -\infty} \int_t^{\frac{3}{2}} \frac{1}{(2x)^2+9} dx = \lim_{t \rightarrow -\infty} \left[ \frac{1}{6} \arctan\left(\frac{2x}{3}\right) \right]_t^{\frac{3}{2}} \\ &= \lim_{t \rightarrow -\infty} \left[ \frac{1}{6} \left( \frac{\pi}{4} \right) - \frac{1}{6} \arctan\left(\frac{2t}{3}\right) \right] = \frac{\pi}{24} - \frac{1}{6} \left( -\frac{\pi}{2} \right) = \frac{\pi}{24} + \frac{\pi}{12} = \frac{\pi}{8}.\end{aligned}$$

So the integral converges.

(k) Using integration by parts with  $w = x$  and  $dv = e^{-x} dx$ , we get

$$\begin{aligned}\int_0^\infty x e^{-x} dx &= \lim_{t \rightarrow \infty} \int_0^t x e^{-x} dx = \lim_{t \rightarrow \infty} \left( [-x e^{-x}]_0^t + \int_0^t e^{-x} dx \right) \\ &= \lim_{t \rightarrow \infty} [-x e^{-x} - e^{-x}]_0^t = \lim_{t \rightarrow \infty} (-t e^{-t} - e^{-t} + 1) \\ &= \lim_{t \rightarrow \infty} \frac{-t}{e^t} - \lim_{t \rightarrow \infty} e^{-t} + \lim_{t \rightarrow \infty} 1 = \lim_{t \rightarrow \infty} \frac{-t}{e^t} - 0 + 1 \\ &\stackrel{H}{=} \lim_{t \rightarrow \infty} \frac{-1}{e^t} + 1 = 0 + 1 = 1.\end{aligned}$$

So the integral converges.

(ℓ) Using integration by parts with  $w = \ln(x)$  and  $dv = \frac{1}{x\sqrt{x}} dx$ , we obtain

$$\begin{aligned}\int_1^\infty \frac{\ln(x)}{x\sqrt{x}} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{\ln(x)}{x\sqrt{x}} dx = \lim_{t \rightarrow \infty} \left( \left[ -\frac{2\ln(x)}{\sqrt{x}} \right]_1^t + \int_1^t \frac{2}{x\sqrt{x}} dx \right) \\ &= \lim_{t \rightarrow \infty} \left[ -\frac{2\ln(x)}{\sqrt{x}} - \frac{4}{\sqrt{x}} \right]_1^t = \lim_{t \rightarrow \infty} \left( -\frac{2\ln(t)}{\sqrt{t}} - \frac{4}{\sqrt{t}} + 4 \right) \\ &= \lim_{t \rightarrow \infty} \frac{-2\ln(t)}{\sqrt{t}} - \lim_{t \rightarrow \infty} \frac{4}{\sqrt{t}} + \lim_{t \rightarrow \infty} 4 = \lim_{t \rightarrow \infty} \frac{-2\ln(t)}{\sqrt{t}} - 0 + 4 \\ &\stackrel{\text{H}}{=} \lim_{t \rightarrow \infty} \frac{-\frac{2}{t}}{\frac{1}{2\sqrt{t}}} + 4 = \lim_{t \rightarrow \infty} \frac{-4}{\sqrt{t}} + 4 = -0 + 4 = 4.\end{aligned}$$

So the integral converges.