MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Assignment 7 — **MATHEMATICS 1001** — Fall 2019

SOLUTIONS

[5] 1. (a) First observe that

$$x^3 - 3x^2 + 4x - 12 = x^2(x-3) + 4(x-3) = (x-3)(x^2+4),$$

so the form of the partial fraction decomposition is

$$\frac{2x+3}{x^3 - 3x^2 + 4x - 12} = \frac{A}{x-3} + \frac{B+C}{x^2+4}.$$

Multiplying both sides by the original denominator, we have

$$2x + 3 = A(x^2+4) + (Bx+C)(x-3).$$

When $x = 3$, we have

$$9 = 13A \quad \Longrightarrow \quad A = \frac{9}{13}.$$

When $x = 0$, we have

$$3 = 4A - 3C = \frac{36}{13} - 3C \quad \Longrightarrow \quad 3C = -\frac{3}{13} \quad \Longrightarrow \quad C = -\frac{1}{13}.$$

And when, say, $x = 1$, we have

$$5 = 5A - 2(B+C) = \frac{45}{13} - 2B + \frac{2}{13} \quad \Longrightarrow \quad 2B = -\frac{18}{13} \quad \Longrightarrow \quad B = -\frac{9}{13}.$$

Thus the integral becomes

$$\begin{aligned}
\int \frac{2x+3}{x^3 - 3x^2 + 4x - 12}\,dx &= \int \left[\frac{\frac{9}{13}}{x-3} + \frac{-\frac{9}{13}x - \frac{1}{13}}{x^2+4}\right] dx \\
&= \int \left[\frac{9}{13}\cdot\frac{1}{x-3} - \frac{9}{13}\cdot\frac{x}{x^2+4} - \frac{1}{13}\cdot\frac{1}{x^2+4}\right] dx \\
&= \frac{9}{13}\ln|x-3| - \frac{1}{26}\arctan\left(\frac{x}{2}\right) - \frac{9}{13}\int \frac{x}{x^2+4}\,dx.
\end{aligned}$$

For the remaining integral, we let $u = x^2 + 4$ so $du = 2x\,dx$ and $\frac{1}{2}\,du = x\,dx$. Thus

$$\int \frac{x}{x^2+4}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+4) + C$$

and so

$$\int \frac{2x+3}{x^3 - 3x^2 + 4x - 12}\,dx = \frac{9}{13}\ln|x-3| - \frac{1}{26}\arctan\left(\frac{x}{2}\right) - \frac{9}{26}\ln(x^2+4) + C.$$

[5] (b) Observe that

$$x^5 - 4x^4 + 4x^3 = x^3(x^2 - 4x + 4) = x^3(x-2)^2$$

so the form of the partial fraction decomposition is

$$\frac{5x^4 + 32x - 32}{x^5 - 4x^4 + 4x^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-2} + \frac{E}{(x-2)^2}.$$

Multiplying both sides by the original denominator, we obtain

$$\begin{aligned} 5x^4 + 32x - 32 &= Ax^2(x-2)^2 + Bx(x-2)^2 + C(x-2)^2 + Dx^3(x-2) + Ex^3 \\ &= (A+D)x^4 + (-4A + B - 2D + E)x^3 + (4A - 4B + C)x^2 \\ &\quad + (4B - 4C)x + 4C. \end{aligned}$$

Comparing like coefficients, we immediately have

$$4C = -32 \quad \Longrightarrow \quad C = -8.$$

Next,

$$4B - 4C = 32 \quad \Longrightarrow \quad 4B + 32 = 32 \quad \Longrightarrow \quad 4B = 0 \quad \Longrightarrow \quad B = 0.$$

Then

$$4A - 4B + C = 0 \quad \Longrightarrow \quad 4A - 0 - 8 = 0 \quad \Longrightarrow \quad 4A = 8 \quad \Longrightarrow \quad A = 2$$

and

$$A + D = 5 \quad \Longrightarrow \quad 2 + D = 5 \quad \Longrightarrow \quad D = 3.$$

Finally,

$$-4A + B - 2D + E = 0 \quad \Longrightarrow \quad -8 + 0 - 6 + E = 0 \quad \Longrightarrow \quad E = 14.$$

So we can write the integral as

$$\begin{aligned} \int \frac{5x^4 + 32x - 32}{x^5 - 4x^4 + 4x^3}\,dx &= \int \left[\frac{2}{x} - \frac{8}{x^2} + \frac{3}{x-2} + \frac{14}{(x-2)^2}\right] dx \\ &= 2\ln|x| + \frac{8}{x} + 3\ln|x-2| - \frac{14}{x-2} + C. \end{aligned}$$

[5] (c) The integrand is an improper rational function, so first we need to carry out long division and find that

$$\frac{4x^4 - 8x^3 - 21x^2}{4x^2 - 1} = x^2 - 2x - 5 + \frac{-2x + 5}{4x^2 - 1}.$$

Now we can factor and decompose the resulting proper rational function:

$$\frac{-2x+5}{4x^2-1} = \frac{-2x+5}{(2x-1)(2x+1)} = \frac{A}{2x-1} + \frac{B}{2x+1}$$

$$-2x + 5 = A(2x+1) + B(2x-1).$$

When $x = \frac{1}{2}$ we have

$$4 = 2A \implies A = 2$$

and when $x = -\frac{1}{2}$ this becomes

$$6 = -2A \implies A = -3.$$

Hence the integral becomes

$$\int \frac{4x^4 - 8x^3 - 21x^2}{4x^2 - 1}\,dx = \int \left[x^2 - 2x - 5 + \frac{2}{2x-1} - \frac{3}{2x+1}\right] dx$$
$$= \frac{1}{3}x^3 - x^2 - 5x + \ln|2x-1| - \frac{3}{2}\ln|2x+1| + C.$$

[5] (d) The denominator is already factored, so we can immediately write

$$\frac{3x^4 + 4x^3 + 6x^2 + 3}{(x^2+1)^3} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{Ex+F}{(x^2+1)^3}$$
$$3x^4 + 4x^3 + 6x^2 + 3 = (Ax+B)(x^2+1)^2 + (Cx+D)(x^2+1) + Ex + F$$
$$= Ax^5 + Bx^4 + (2A+C)x^3 + (2B+D)x^2$$
$$+ (A+C+E)x + (B+D+F).$$

Comparing like coefficients, right away we have $A = 0$ and $B = 3$. Then

$$2A + C = 4 \implies 0 + C = 4 \implies C = 4$$
$$2B + D = 6 \implies 6 + D = 6 \implies D = 0$$
$$A + C + E = 0 \implies 0 + 4 + E = 0 \implies E = -4$$
$$B + D + F = 3 \implies 3 + 0 + F = 3 \implies F = 0.$$

So the integral becomes

$$\int \frac{3x^4 + 4x^3 + 6x^2 + 3}{(x^2+1)^3}\,dx = \int \left[\frac{3}{x^2+1} + \frac{4x}{(x^2+1)^2} - \frac{4x}{(x^2+1)^3}\right] dx$$
$$= 3\arctan(x) + 4\int \frac{x}{(x^2+1)^2}\,dx - 4\int \frac{x}{(x^2+1)^3}\,dx.$$

For the two remaining integrals, let $u = x^2 + 1$ so $du = 2x\,dx$ and $\frac{1}{2}\,du = x\,dx$. Then

$$\int \frac{x}{(x^2+1)^2}\,dx = \frac{1}{2}\int \frac{1}{u^2}\,du = -\frac{1}{2u} + C = -\frac{1}{2(x^2+1)} + C$$

and similarly

$$\int \frac{x}{(x^2+1)^3}\,dx = \frac{1}{2}\int \frac{1}{u^3}\,du = -\frac{1}{4u^2} + C = -\frac{1}{4(x^2+1)^2} + C.$$

So finally,

$$\int \frac{3x^4 + 4x^3 + 6x^2 + 3}{(x^2+1)^3}\,dx = 3\arctan(x) - \frac{2}{x^2+1} + \frac{1}{(x^2+1)^2} + C.$$

[5] 2. (a) Since the power of $\sin(x)$ is odd, we set aside one factor of $\sin(x)$ for u-substitution and write

$$\int \frac{\sin^7(x)}{\sqrt{\cos(x)}}\,dx = \int \frac{\sin^6(x)}{\sqrt{\cos(x)}}\cdot \sin(x)\,dx = \int \frac{[1-\cos^2(x)]^3}{\sqrt{\cos(x)}}\cdot \sin(x)\,dx.$$

Let $u = \cos(x)$ so $du = -\sin(x)\,dx$ and $-\,dx = \sin(x)\,dx$. Then the integral becomes

$$\begin{aligned}\int \frac{\sin^7(x)}{\sqrt{\cos(x)}}\,dx &= -\int \frac{[1-u^2]^3}{\sqrt{u}}\,du \\ &= -\int \frac{1-3u^2+3u^4-u^6}{\sqrt{u}}\,du \\ &= -\int \left[u^{-\frac{1}{2}} - 3u^{\frac{3}{2}} + 3u^{\frac{7}{2}} - u^{\frac{11}{2}}\right] du \\ &= -\left[2\sqrt{u} - \frac{6}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{9}{2}} - \frac{2}{13}u^{\frac{13}{2}}\right] + C \\ &= -2\sqrt{\cos(x)} + \frac{6}{5}[\cos(x)]^{\frac{5}{2}} - \frac{2}{3}[\cos(x)]^{\frac{9}{2}} + \frac{2}{13}[\cos(x)]^{\frac{13}{2}} + C.\end{aligned}$$

[5] (b) Using the half-angle formulas, we write

$$\begin{aligned}\int \cos^4(3x)\,dx &= \int [\cos^2(3x)]^2\,dx \\ &= \int \left[\frac{1+\cos(6x)}{2}\right]^2 dx \\ &= \frac{1}{4}\int [1 + 2\cos(6x) + \cos^2(6x)]\,dx \\ &= \frac{1}{4}\int \left[1 + 2\cos(6x) + \frac{1+\cos(12x)}{2}\right] dx \\ &= \frac{1}{4}\int \left[\frac{3}{2} + 2\cos(6x) + \frac{1}{2}\cos(12x)\right] dx \\ &= \frac{1}{4}\left[\frac{3}{2}x + \frac{1}{3}\sin(6x) + \frac{1}{24}\sin(12x)\right] + C \\ &= \frac{3}{8}x + \frac{1}{12}\sin(6x) + \frac{1}{96}\sin(12x) + C.\end{aligned}$$

[5] (c) Although this is written as a quotient of sine and cosine functions with even powers, our

standard approach for this case doesn't work. So instead we write

$$\begin{aligned}
\int \frac{\sin^4(x)}{\cos^4(x)}\,dx &= \int \tan^4(x)\,dx \\
&= \int \tan^2(x)\tan^2(x)\,dx \\
&= \int \tan^2(x)[\sec^2(x)-1]\,dx \\
&= \int [\tan^2(x)\sec^2(x)-\tan^2(x)]\,dx \\
&= \int [\tan^2(x)\sec^2(x)-\sec^2(x)+1]\,dx \\
&= -\tan(x)+x+\int \tan^2(x)\sec^2(x)\,dx.
\end{aligned}$$

For this remaining integral, we let $u=\tan(x)$ so $du=\sec^2(x)\,dx$, and hence

$$\int \tan^2(x)\sec^2(x)\,dx = \int u^2\,du = \frac{1}{3}u^3+C = \frac{1}{3}\tan^3(x)+C.$$

Therefore

$$\int \frac{\sin^4(x)}{\cos^4(x)}\,dx = -\tan(x)+x+\frac{1}{3}\tan^3(x)+C.$$

[5] (d) First we let $u=\sqrt{x}$ so $du=\dfrac{1}{2\sqrt{x}}\,dx$ and $2\,du=\dfrac{1}{\sqrt{x}}\,dx$. The integral becomes

$$\int \frac{\cot^3\left(\sqrt{x}\right)\csc^9\left(\sqrt{x}\right)}{\sqrt{x}}\,dx = 2\int \cot^3(u)\csc^9(u)\,du.$$

Since the power of $\csc(u)$ is odd, we extract a factor of $\cot(u)\csc(u)$ for substitution purposes, and write

$$\begin{aligned}
\int \frac{\cot^3\left(\sqrt{x}\right)\csc^8\left(\sqrt{x}\right)}{\sqrt{x}}\,dx &= 2\int \cot^2(u)\csc^8(u)\cdot\csc(u)\cot(u)\,du \\
&= 2\int [\csc^2(u)-1]\csc^8(u)\cdot\csc(u)\cot(u)\,du.
\end{aligned}$$

Now we perform another substitution, letting $v=\csc(u)$ so $dv=-\csc(u)\cot(u)\,du$ and

$-\,dv = \csc(u)\cot(u)\,du$. Thus we have

$$\begin{aligned}
\int \frac{\cot^3\left(\sqrt{x}\right)\csc^8\left(\sqrt{x}\right)}{\sqrt{x}}\,dx &= -2\int [v^2-1]v^8\,dv \\
&= -2\int [v^{10}-v^8]\,dv \\
&= -2\left[\frac{1}{11}v^{11}-\frac{1}{9}v^9\right]+C \\
&= -\frac{2}{11}\csc^{11}(u)+\frac{2}{9}\csc^9(u)+C \\
&= -\frac{2}{11}\csc^{11}\left(\sqrt{x}\right)+\frac{2}{9}\csc^9\left(\sqrt{x}\right)+C.
\end{aligned}$$