MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Assignment 7	**MATHEMATICS 1001**	Fall 2025

SOLUTIONS

[4] 1. (a) This integral involves a radical of the form $\sqrt{k^2 - x^2}$ with $k = 3$. Thus we let $x = 3\sin(\theta)$ so $dx = 3\cos(\theta)\,d\theta$ and

$$\sqrt{9 - x^2} = \sqrt{9 - 9\sin^2(\theta)} = 3\cos(\theta).$$

The integral becomes

$$\begin{aligned}\int \frac{1}{x\sqrt{9-x^2}}\,dx &= \int \frac{1}{3\sin(\theta)\cdot 3\cos(\theta)} \cdot 3\cos(\theta)\,d\theta \\ &= \frac{1}{3}\int \csc(\theta)\,d\theta \\ &= -\frac{1}{3}\ln|\csc(\theta) + \cot(\theta)| + C.\end{aligned}$$

Now we can draw a right triangle with interior angle θ, opposite side of length x and hypotenuse of length 3. The adjacent sidelength will be $\sqrt{9 - x^2}$ so

$$\csc(\theta) = \frac{3}{x} \quad \text{and} \quad \cot(\theta) = \frac{\sqrt{9-x^2}}{x}.$$

Thus

$$\begin{aligned}\int \frac{1}{x\sqrt{9-x^2}}\,dx &= -\frac{1}{3}\ln\left|\frac{3}{x} + \frac{\sqrt{9-x^2}}{x}\right| + C \\ &= -\frac{1}{3}\ln\left|\frac{3+\sqrt{9-x^2}}{x}\right| + C.\end{aligned}$$

[5] (b) This is a $\sqrt{x^2 - k^2}$ type radical with $k = 1$, so we let $x = \sec(\theta)$ and $dx = \sec(\theta)\tan(\theta)\,d\theta$. Then

$$\sqrt{x^2 - 1} = \sqrt{\sec^2(\theta) - 1} = \tan(\theta).$$

When $x = \sqrt{2}$, we have

$$\sqrt{2} = \sec(\theta) \implies \frac{\sqrt{2}}{2} = \cos(\theta) \implies \theta = \frac{\pi}{4}.$$

When $x = 2$, we have

$$2 = \sec(\theta) \implies \frac{1}{2} = \cos(\theta) \implies \theta = \frac{\pi}{3}.$$

Hence the integral becomes

$$\int_{\sqrt{2}}^{2} \frac{24}{x^3\sqrt{x^2-1}}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{24}{\sec^3(\theta)\tan(\theta)} \cdot \sec(\theta)\tan(\theta)\,d\theta$$
$$= 24\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sec^2(\theta)}\,d\theta$$
$$= 24\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^2(\theta)\,d\theta$$

Now we use the half-angle formula for cosine to obtain

$$\int_{\sqrt{2}}^{2} \frac{24}{x^3\sqrt{x^2-1}}\,dx = 24\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1+\cos(2\theta)}{2}\,d\theta$$
$$= 12\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} [1+\cos(2\theta)]\,d\theta$$
$$= 12\left[\theta + \frac{1}{2}\sin(2\theta)\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$
$$= 12\left[\left(\frac{\pi}{3} + \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right) - \left(\frac{\pi}{4} + \frac{1}{2}\cdot 1\right)\right]$$
$$= \pi + 3\sqrt{3} - 8.$$

[5] 2. First we write

$$\int_0^{\infty} \frac{e^x}{e^{2x}+1}\,dx = \lim_{T\to\infty}\int_0^{T} \frac{e^x}{e^{2x}+1}\,dx = \lim_{T\to\infty}\int_0^{T} \frac{e^x}{(e^x)^2+1}\,dx.$$

Now let $u = e^x$ so $du = e^x\,dx$. When $x = 0$, $u = 1$, and when $x = T$, $u = e^T$. Thus the integral becomes

$$\int_0^{\infty} \frac{e^x}{e^{2x}+1}\,dx = \lim_{T\to\infty}\int_1^{e^T} \frac{1}{u^2+1}\,du$$
$$= \lim_{T\to\infty}\Big[\arctan(u)\Big]_1^{e^T}$$
$$= \lim_{T\to\infty}[\arctan(e^T) - \arctan(1)]$$
$$= \frac{\pi}{2} - \frac{\pi}{4}$$
$$= \frac{\pi}{4}.$$

[3] 3. This is a separable equation, because it can be rewritten as

$$\begin{aligned} t^3 \frac{dy}{dt} &= 8\sec(y) \\ \frac{1}{\sec(y)}\, dy &= \frac{8}{t^3}\, dt \\ \int \cos(y)\, dy &= 8 \int t^{-3}\, dt \\ \sin(y) &= 8 \cdot \frac{t^{-2}}{-2} + C \\ \sin(y) &= C - \frac{4}{t^2}. \end{aligned}$$

This is the general solution, which we could also express in an explicit form as

$$y = \arcsin\left(C - \frac{4}{t^2}\right).$$

Now we can use the fact that $y = \frac{\pi}{2}$ when $t = 1$, so

$$\begin{aligned} \sin\left(\frac{\pi}{2}\right) &= C - \frac{4}{1^2} \\ 1 &= C - 4 \\ C &= 5. \end{aligned}$$

Hence the particular solution is

$$\boxed{\sin(y) = 5 - \frac{4}{t^2}} \quad \text{or} \quad \boxed{y = \arcsin\left(5 - \frac{4}{t^2}\right).}$$

[3] 4. If y is the number of caribou remaining after t years, we know that

$$y = y_0 e^{kt}.$$

If we assume that we want to find $y(0) = y_0$, we are given that $y(2) = 200$ and $y(3) = 120$. Thus

$$y_0 e^{2k} = 200 \quad \text{and} \quad y_0 e^{3k} = 120.$$

Dividing the first of these equations into the second, we obtain

$$\begin{aligned} \frac{y_0 e^{3k}}{y_0 e^{2k}} &= \frac{120}{200} \\ e^{3k-2k} &= \frac{3}{5} \\ e^k &= \frac{3}{5} \\ k &= \ln\left(\frac{3}{5}\right). \end{aligned}$$

Hence

$$y = y_0 e^{t \ln\left(\frac{3}{5}\right)}$$

and we can use either given piece of information to determine y_0. For instance, if we use the fact that $y(2) = 200$, we have

$$\begin{aligned} y_0 e^{2 \ln\left(\frac{3}{5}\right)} &= 200 \\ y_0 e^{\ln\left(\frac{9}{25}\right)} &= 200 \\ y_0 \cdot \frac{9}{25} &= 200 \\ y_0 &= \frac{5000}{9} \approx 556. \end{aligned}$$

Hence there were about 556 caribou two years before the start of the study.