MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

INTEGRATION STRATEGIES **Math 1001 Worksheet** FALL 2019

SOLUTIONS

1. (a) We use u-substitution, with $u = x^2 + 4$ so $\frac{1}{2}\,du = x\,dx$. Then

$$\int \frac{x}{\sqrt{x^2+4}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$$
$$= \frac{1}{2}\left[2\sqrt{u}\right] + C$$
$$= \sqrt{x^2+4} + C.$$

(b) We use trigonometric substitution, with $x = 2\sin(\theta)$. Then $dx = 2\cos(\theta)\,d\theta$ and $\sqrt{4-x^2} = 2\cos(\theta)$. The integral becomes

$$\int \frac{\sqrt{4-x^2}}{x}\,dx = \int \frac{2\cos(\theta)}{2\sin(\theta)} \cdot 2\cos(\theta)\,d\theta$$
$$= 2\int \frac{\cos^2(\theta)}{\sin(\theta)}\,d\theta$$
$$= 2\int \frac{1-\sin^2(\theta)}{\sin(\theta)}\,d\theta$$
$$= 2\int [\csc(\theta) - \sin(\theta)]\,d\theta$$
$$= -2\ln|\csc(\theta) + \cot(\theta)| + 2\cos(\theta) + C.$$

We already know that $\sin(\theta) = \frac{1}{2}x$, so $\csc(\theta) = \frac{2}{x}$. Furthermore, $\cos(\theta) = \frac{1}{2}\sqrt{4-x^2}$. Finally,

$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)} = \frac{\frac{1}{2}\sqrt{4-x^2}}{\frac{1}{2}x} = \frac{\sqrt{4-x^2}}{x}.$$

Hence

$$\int \frac{\sqrt{4-x^2}}{x}\,dx = -2\ln\left|\frac{2}{x} + \frac{\sqrt{4-x^2}}{x}\right| + 2\left(\frac{1}{2}\sqrt{4-x^2}\right) + C$$
$$= -2\ln\left|\frac{\sqrt{4-x^2}+2}{x}\right| + \sqrt{4-x^2} + C.$$

(c) We complete the square:

$$\begin{aligned} 4x^2 - 12x + 13 &= 4\left[x^2 - 3x + \frac{13}{4}\right] \\ &= 4\left[\left(x^2 - 3x + \frac{9}{4}\right) + 1\right] \\ &= 4\left(x - \frac{3}{2}\right)^2 + 4 \\ &= (2x-3)^2 + 4. \end{aligned}$$

Thus

$$\int \frac{1}{4x^2 - 12x + 13}\, dx = \int \frac{1}{(2x-3)^2 + 4}\, dx.$$

Let $u = 2x - 3$ so $\frac{1}{2}\, du = dx$. The integral becomes

$$\begin{aligned} \frac{1}{2}\int \frac{1}{u^2+4}\, du &= \frac{1}{2}\cdot\frac{1}{2}\arctan\left(\frac{u}{2}\right) + C \\ &= \frac{1}{4}\arctan\left(\frac{2x-3}{2}\right) + C. \end{aligned}$$

(d) We can factor the denominator:

$$4x^2 - 4x - 3 = (2x+1)(2x-3).$$

Thus an appropriate partial fraction decomposition is

$$\begin{aligned} \frac{1}{4x^2 - 4x - 3} &= \frac{A}{2x+1} + \frac{B}{2x-3} \\ 1 &= A(2x-3) + B(2x+1). \end{aligned}$$

When $x = -\frac{1}{2}$, we get $1 = -4A$ so $A = -\frac{1}{4}$. When $x = \frac{3}{2}$, we get $1 = 4B$ so $B = \frac{1}{4}$. Thus

$$\begin{aligned} \frac{1}{4x^2 - 4x - 3}\, dx &= \int \left(\frac{-\frac{1}{4}}{2x+1} + \frac{\frac{1}{4}}{2x-3}\right) dx \\ &= -\frac{1}{4}\cdot\frac{1}{2}\ln|2x+1| + \frac{1}{4}\cdot\frac{1}{2}\ln|2x-3| + C \\ &= \frac{1}{8}\ln|2x-3| - \frac{1}{8}\ln|2x+1| + C. \end{aligned}$$

(e) We use u-substitution, with $u = \ln(x)$ so $du = \frac{1}{x}\, dx$. The integral becomes

$$\begin{aligned} \int \frac{1}{x[\ln(x)]^2}\, dx &= \int \frac{1}{u^2}\, du \\ &= -\frac{1}{u} + C \\ &= -\frac{1}{\ln(x)} + C. \end{aligned}$$

(f) We use integration by parts. Let $w = [\ln(x)]^2$ so $dw = \dfrac{2\ln(x)}{x}\,dx$. Let $dv = x\,dx$ so $v = \frac{1}{2}x^2$. Then

$$\int x[\ln(x)]^2\,dx = \frac{1}{2}x^2[\ln(x)]^2 - \int x\ln(x)\,dx.$$

We need to use integration by parts a second time. Let $w = \ln(x)$ so $dw = \frac{1}{x}\,dx$, and let $dv = x\,dx$ so $v = \frac{1}{2}x^2$. Now

$$\begin{aligned}\int x[\ln(x)]^2\,dx &= \frac{1}{2}x^2[\ln(x)]^2 - \left[\frac{1}{2}x^2\ln(x) - \frac{1}{2}\int x\,dx\right]\\ &= \frac{1}{2}x^2[\ln(x)]^2 - \frac{1}{2}x^2\ln(x) + \frac{1}{4}x^2 + C.\end{aligned}$$

(g) We use a trig integration strategy. Since the powers of $\tan(x)$ and $\sec(x)$ are both even, we factor out a $\sec^2(x)$ and then use the identity

$$\sec^2(x) = \tan^2(x) + 1$$

to obtain

$$\begin{aligned}\int \tan^2(x)\sec^4(x)\,dx &= \int \tan^2(x)\sec^2(x)\sec^2(x)\,dx\\ &= \int \tan^2(x)[\tan^2(x)+1]\sec^2(x)\,dx\\ &= \int [\tan^4(x)+\tan^2(x)]\sec^2(x)\,dx.\end{aligned}$$

Now let $u = \tan(x)$ so $du = \sec^2(x)\,dx$. The integral becomes

$$\begin{aligned}\int [u^4+u^2]\,du &= \frac{1}{5}u^5 + \frac{1}{3}u^3 + C\\ &= \frac{1}{5}\tan^5(x) + \frac{1}{3}\tan^3(x) + C.\end{aligned}$$

(h) Here we simply have to realise that

$$\tan^2(x)\cot^2(x) = \tan^2(x)\cdot\frac{1}{\tan^2(x)} = 1,$$

so

$$\int \tan^2(x)\cot^2(x)\,dx = \int dx = x + C.$$

(i) Note that

$$\tan^2(x)\cos^2(x) = \frac{\sin^2(x)}{\cos^2(x)}\cdot\cos^2(x) = \sin^2(x).$$

Remember that we have to use the half-angle formula in order to integrate $\sin^2(x)$, so we write

$$\begin{aligned}\int \tan^2(x)\cos^2(x)\,dx &= \int \sin^2(x) \\ &= \frac{1}{2}\int [1-\cos(2x)]\,dx \\ &= \frac{1}{2}x - \frac{1}{4}\sin(2x) + C.\end{aligned}$$

(j) This is trickier. First note that this is not an elementary integral, and we can rule out most techniques: u-substitution, trig substitution, partial fractions, trig integrals. That pretty much means that we must be able to use integration by parts. However, as written, parts doesn't get us anywhere: there's no choice of w and dv that will provide a manageable integral. So, instead, let's rewrite the integral as

$$\int x[\sec^2(x)-1]\,dx = \int x\sec^2(x)\,dx - \int x\,dx = \int x\sec^2(x)\,dx - \frac{1}{2}x^2.$$

For the remaining integral, parts is a viable option. Let $w = x$ so $dw = dx$. Let $dv = \sec^2(x)\,dx$ so $v = \tan(x)$. Then

$$\begin{aligned}\int x\sec^2(x)\,dx &= x\tan(x) - \int \tan(x)\,dx \\ &= x\tan(x) + \ln|\cos(x)| + C.\end{aligned}$$

Finally, then,

$$\int x\tan^2(x)\,dx = x\tan(x) + \ln|\cos(x)| - \frac{1}{2}x^2 + C.$$