MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 6 **MATHEMATICS 1001** WINTER 2015

SOLUTIONS

[4] 1. (a) Let $u = \tan^2(x)$ so $f(u) = \int_4^u e^{t^3}\,dt$. Then, by the Chain Rule,

$$f'(x) = f'(u)u' = e^{u^3} \cdot 2\tan(x)\sec^2(x) = 2e^{\tan^6(x)}\tan(x)\sec^2(x).$$

[4] (b) We write

$$\int_{-x}^{\sqrt{x}} \cos(t^2)\,dt = \int_{-x}^{0} \cos(t^2)\,dt + \int_{0}^{\sqrt{x}} \cos(t^2)\,dt = -\int_{0}^{-x} \cos(t^2)\,dt + \int_{0}^{\sqrt{x}} \cos(t^2)\,dt.$$

Then

$$f'(x) = -\cos((-x)^2)\cdot[-x]' + \cos\left(\left(\sqrt{x}\right)^2\right)\cdot\left[\sqrt{x}\right]' = \cos(x^2) + \frac{\cos(x)}{2\sqrt{x}}.$$

[2] 2. (a) Observe that the line $y = \frac{1}{2}x + 1$ moves upwards to the right, with intercepts $(0,1)$ and $(-2,0)$. The curve $y = \sqrt{x+2}$ is a semi-parabola, opening upwards to the right, with vertex $(-2,0)$. To find any points of intersection, we set

$$\begin{aligned}\frac{1}{2}x + 1 &= \sqrt{x+2}\\ \frac{1}{4}x^2 + x + 1 &= x + 2\\ \frac{1}{4}x^2 - 1 &= 0\\ \frac{1}{4}(x-2)(x+2) &= 0\end{aligned}$$

and so $x = -2$ or $x = 2$ (as can be verified by substitution back into the original expressions). Thus the points of intersection are $(-2,0)$ and $(2,2)$, and we obtain the graph found in Figure 1.

[5] (b) From the graph, we can see that the top boundary curve of the region is the semi-parabola $y = \sqrt{x+2}$, while the bottom boundary curve is the line $y = \frac{1}{2}x + 1$. The left- and right-endpoints of the region are $x = -2$ and $x = 2$. Thus the area is

$$\begin{aligned}A &= \int_{-2}^{2}\left[\sqrt{x+2} - \left(\frac{1}{2}x + 1\right)\right]dx\\ &= \int_{-2}^{2}\left[\sqrt{x+2} - \frac{1}{2}x - 1\right]dx\\ &= \left[\frac{2}{3}(x+2)^{\frac{3}{2}} - \frac{1}{4}x^2 - x\right]_{-2}^{2}\\ &= \frac{4}{3}.\end{aligned}$$

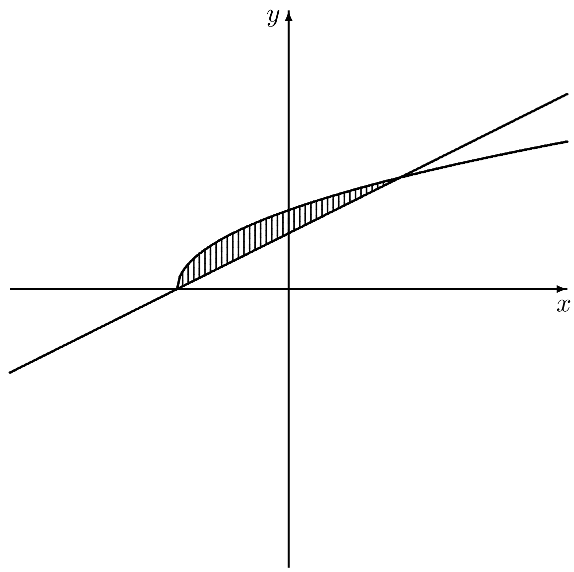

Figure 1: Question 2(a)

[5] (c) In terms of functions of y, we can rewrite the line as $x = 2y - 2$ and the semi-parabola as $x = y^2 - 2$ (where $y \geq 0$). The righthand boundary curve is the line, while the lefthand boundary curve is the the semi-parabola. The bottom and top endpoints of the region are $y = 0$ and $y = 2$. Hence the area is given by

$$\begin{aligned} A &= \int_0^2 [(2y-2) - (y^2-2)]\,dy \\ &= \int_0^2 [2y - y^2]\,dy \\ &= \left[y^2 - \frac{1}{3}y^3\right]_0^2 \\ &= \frac{4}{3}. \end{aligned}$$

As expected, we obtain the same answer for both parts (b) and (c).

[5] 3. (a) Figure 2 gives a sketch of the region. To find the points of intersection, we set

$$\begin{aligned} 4 - x^2 &= 2x^2 + x - 6 \\ 3x^2 + x - 10 &= 0 \\ (3x-5)(x+2) &= 0 \end{aligned}$$

so $x = \frac{5}{3}$ or $x = -2$. From the graph of by substituting, say, $x = 0$ into each expression, we can determine that the top boundary curve is $y = 4 - x^2$ and the bottom boundary curve is $y = 2x^2 + x - 6$. Thus

$$\begin{aligned} A &= \int_{-2}^{\frac{5}{3}} [(4 - x^2) - (2x^2 + x - 6)]\, dx \\ &= \int_{-2}^{\frac{5}{3}} [10 - x - 3x^2]\, dx \\ &= \left[10x - \frac{1}{2}x^2 - x^3\right]_{-2}^{\frac{5}{3}} \\ &= \frac{1331}{54}. \end{aligned}$$

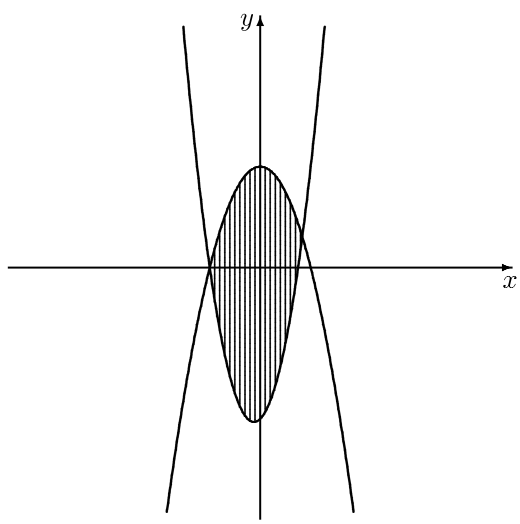

Figure 2: Question 3(a)

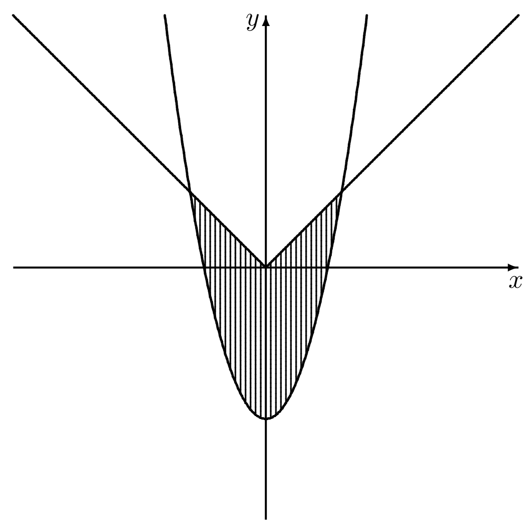

Figure 3: Question 3(b)

[5] (b) This region is illustrated in Figure 3. In order to find the points of intersection, we must recall that

$$|x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0. \end{cases}$$

Thus we first set

$$\begin{aligned} x &= x^2 - 6 \\ x^2 - x - 6 &= 0 \\ (x - 3)(x + 2) &= 0 \end{aligned}$$

so $x = 3$ or $x = -2$. However, substitution into $y = |x|$ and $y = x^2 - 6$ reveals that only $x = 3$ is actually a point of intersection. Similarly, we set

$$\begin{aligned} -x &= x^2 - 6 \\ x^2 + x - 6 &= 0 \\ (x + 3)(x - 2) &= 0 \end{aligned}$$

so $x = -3$ or $x = 2$. This time, substitution finds that only $x = -3$ is a point of intersection. Furthermore, from the graph (or by evaluating the two functions at a point such as $x = 0$) we can see that $y = |x|$ is the top boundary curve and $y = x^2 - 6$ is the bottom boundary curve. So

$$\begin{aligned} A &= \int_{-3}^{3} [|x| - (x^2 - 6)]\,dx \\ &= \int_{-3}^{0} [-x - (x^2 - 6)]\,dx + \int_{0}^{3} [x - (x^2 - 6)]\,dx \\ &= \int_{-3}^{0} [-x - x^2 + 6]\,dx + \int_{0}^{3} [x - x^2 + 6]\,dx \\ &= \left[-\frac{1}{2}x^2 - \frac{1}{3}x^3 + 6x\right]_{-3}^{0} + \left[\frac{1}{2}x^2 - \frac{1}{3}x^3 + 6x\right]_{0}^{3} \\ &= \frac{27}{2} + \frac{27}{2} \\ &= 27. \end{aligned}$$

[5] (c) This region is given in Figure 4. To find the points of intersection, we set

$$\begin{aligned} \sin(2x) &= \cos(x) \\ 2\sin(x)\cos(x) &= \cos(x) \\ \cos(x)[2\sin(x) - 1] &= 0. \end{aligned}$$

If $\cos(x) = 0$ then $x = \arccos(0) = \frac{\pi}{2}$. If $2\sin(x) - 1 = 0$ then $x = \arcsin\left(\frac{1}{2}\right)$ so $x = \frac{\pi}{6}$. (There are other intersection points — for example, $\cos(x) = 0$ when $x = -\frac{\pi}{2}$ as well — but none of them lie in the given interval.)

Thus we have two intervals to consider. On $\left[0, \frac{\pi}{6}\right]$, $y = \cos(x)$ is the top boundary curve while $y = \sin(2x)$ is the bottom boundary curve. Conversely, on $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$, $y = \sin(2x)$ is the top boundary curve and $y = \cos(x)$ is the bottom boundary curve. So we have

$$\begin{aligned} A &= \int_{0}^{\frac{\pi}{6}} [\cos(x) - \sin(2x)]\,dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} [\sin(2x) - \cos(x)]\,dx \\ &= \left[\sin(x) + \frac{1}{2}\cos(2x)\right]_{0}^{\frac{\pi}{6}} + \left[-\frac{1}{2}\cos(2x) - \sin(x)\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= \frac{1}{4} + \frac{1}{4} \\ &= \frac{1}{2}. \end{aligned}$$

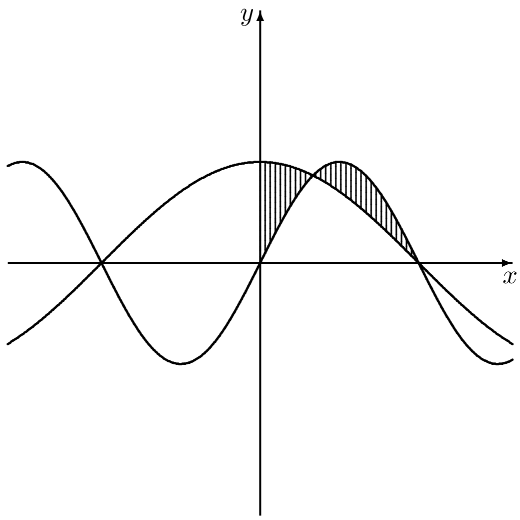

Figure 4: Question 3(c)

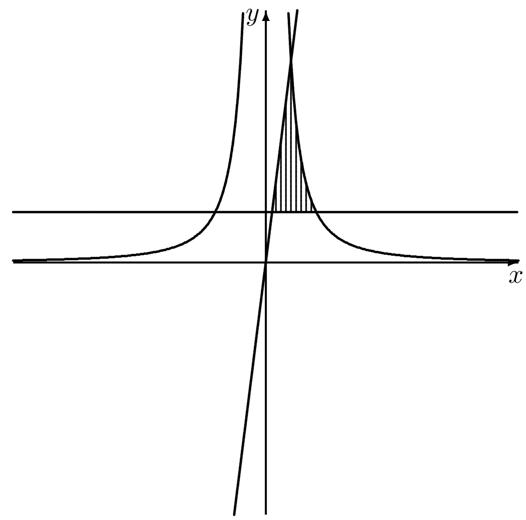

Figure 5: Question 3(d)

[5] (d) The graph of this region can be found in Figure 5. To find any points of intersection, we set

$$\frac{1}{x^2} = 8x$$
$$x^3 = \frac{1}{8}$$
$$x = \frac{1}{2}.$$

Furthermore, the line $y = 8x$ intersects $y = 1$ at $x = \frac{1}{8}$, while the curve $y = \dfrac{1}{x^2}$ intersects $y = 1$ at $x = \pm 1$ (however, $x = -1$ is not a part of the region described in this problem). Given the geometry of the region, we have two choices. We could work in terms of functions of x, in which case we would have to observe that, while $y = 1$ is always the bottom boundary curve, $y = 8x$ is the top boundary curve on the interval $\left[\frac{1}{8}, \frac{1}{2}\right]$ and $y = \dfrac{1}{x^2}$ is the top boundary curve on the interval $\left[\frac{1}{2}, 1\right]$. Thus

$$\begin{aligned}
A &= \int_{\frac{1}{8}}^{\frac{1}{2}} [8x - 1]\, dx + \int_{\frac{1}{2}}^{1} \left[\frac{1}{x^2} - 1\right] \\
&= \left[4x^2 - x\right]_{\frac{1}{8}}^{\frac{1}{2}} + \left[-\frac{1}{x} - x\right]_{\frac{1}{2}}^{1} \\
&= \frac{9}{16} + \frac{1}{2} \\
&= \frac{17}{16}.
\end{aligned}$$

However, it is much easier to work in terms of functions of y. The curve $y = \dfrac{1}{x^2}$ becomes $x = \dfrac{1}{\sqrt{y}}$, and this is always the righthand boundary curve. The line $y = 8x$ is simply $x = \frac{1}{8}y$, and this is always the lefthand boundary curve. We have already found that their point of intersection is $x = \frac{1}{2}$, and by substituting into either expression, we find that its y-coordinate is $y = 4$. Thus

$$\begin{aligned} A &= \int_1^4 \left[\frac{1}{\sqrt{y}} - \frac{1}{8}y\right] dy \\ &= \left[2\sqrt{y} - \frac{1}{16}y^2\right]_1^4 \\ &= \frac{17}{16}. \end{aligned}$$