MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 6 | **MATHEMATICS 1001** | WINTER 2025

SOLUTIONS

[3] 1. (a) A sketch of the region can be found in Figure 1, from which we can observe that it is vertically simple. We can see that, on the interval $[-1, 1]$, $x + 2 \geq x^3 - x$ and so

$$\begin{aligned} A &= \int_{-1}^{1} [(x+2) - (x^3 - x)]\, dx \\ &= \int_{-1}^{1} [2x + 2 - x^3]\, dx \\ &= \left[x^2 + 2x - \frac{1}{4}x^4\right]_{-1}^{1} \\ &= \left[1 + 2 - \frac{1}{4}\right] - \left[1 - 2 - \frac{1}{4}\right] \\ &= 4. \end{aligned}$$

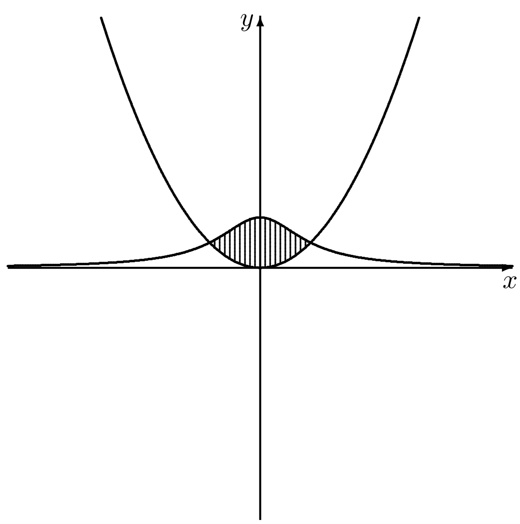

Figure 1: Question 2(a)

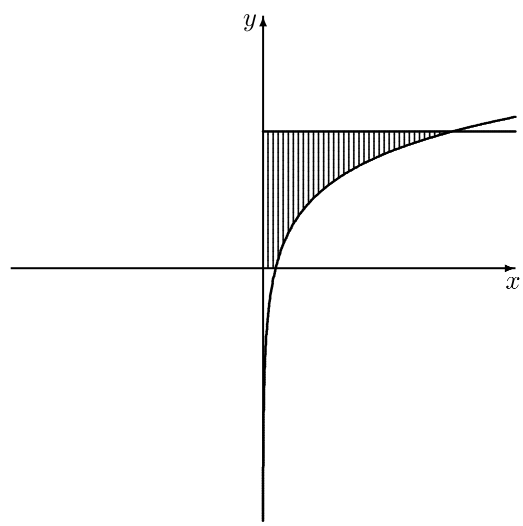

Figure 2: Question 2(b)

[4] (b) This region is illustrated in Figure 2. It is horizontally (but not vertically) simple. Thus we rewrite $y = \ln(x)$ as $x = e^y$ so that the righthand boundary curve is always $f(y) = e^y$ and the lefthand boundary curve is always $x = 0$ (the y-axis). Then

$$\begin{aligned} A &= \int_{0}^{3} [e^y - 0]\, dy \\ &= \left[e^y\right]_{0}^{3} \\ &= e^3 - 1. \end{aligned}$$

[4] 2. First observe that

$$x^4 - 3x^2 - 4 = (x^2 - 4)(x^2 + 1) = (x - 2)(x + 2)(x^2 + 1),$$

so the form of the partial fraction decomposition is

$$\frac{x^3 + x + 30}{x^4 - 3x^2 - 4} = \frac{A}{x - 2} + \frac{B}{x + 2} + \frac{Cx + D}{x^2 + 1}.$$

Multiplying both sides by the original denominator, we have

$$x^3 + x + 30 = A(x + 2)(x^2 + 1) + B(x - 2)(x^2 + 1) + (Cx + D)(x - 2)(x + 2).$$

When $x = 2$, this becomes

$$40 = 20A \quad \Longrightarrow \quad A = 2.$$

When $x = -2$, we have

$$20 = -20B \quad \Longrightarrow \quad B = -1.$$

When $x = 0$, we get

$$30 = 2A - 2B - 4D \quad \Longrightarrow \quad 24 = -4D \quad \Longrightarrow \quad D = -6.$$

And when, say, $x = 1$, we obtain

$$32 = 6A - 2B - 3(C + D) \quad \Longrightarrow \quad 0 = -3D \quad \Longrightarrow \quad C = 0.$$

Thus the integral becomes

$$\int \frac{x^3 + x + 30}{x^4 - 3x^2 - 4}\, dx = \int \left[\frac{2}{x - 2} - \frac{1}{x + 2} - \frac{6}{x^2 + 1}\right] dx$$

$$= 2\ln|x - 2| - \ln|x + 2| - 6\arctan(x) + C.$$

[4] (c) Since the power of $\cos(x)$ is odd, we write

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin(x)}\cos^5(x)\, dx = \int_0^{\frac{\pi}{2}} \sqrt{\sin(x)}\cos^4(x) \cdot \cos(x)\, dx.$$

Now we use the identity $\cos^2(x) = 1 - \sin^2(x)$ and obtain

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin(x)}\cos^5(x)\, dx = \int_0^{\frac{\pi}{2}} \sqrt{\sin(x)}[1 - \sin^2(x)]^2 \cdot \cos(x)\, dx.$$

We let $u = \sin(x)$ so $du = \cos(x)\, dx$. When $x = 0$, $u = 0$. When $x = \frac{\pi}{2}$, $u = 1$. The integral becomes

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sqrt{\sin(x)}\cos^5(x)\,dx &= \int_0^1 \sqrt{u}[1-u^2]^2\,du \\ &= \int_0^1 [\sqrt{u} - 2u^{\frac{5}{2}} + u^{\frac{9}{2}}]\,du \\ &= \left[\frac{2}{3}u^{\frac{3}{2}} - \frac{4}{7}u^{\frac{7}{2}} + \frac{2}{11}u^{\frac{11}{2}}\right]_0^1 \\ &= \frac{2}{3} - \frac{4}{7} + \frac{2}{11} \\ &= \frac{64}{231}.\end{aligned}$$

[5] (d) As given, this isn't immediately recognisable as a trigonometric integral. However, we can let $u = e^x$ so $du = e^x\,dx$, and the integral becomes

$$\int e^x \sin^2(e^x)\cos^2(e^x)\,dx = \int \sin^2(u)\cos^2(u)\,du.$$

Now we can use the half-angle formulas and obtain

$$\begin{aligned}\int e^x \sin^2(e^x)\cos^2(e^x)\,dx &= \int \frac{1-\cos(2u)}{2}\cdot\frac{1+\cos(2u)}{2}\,du \\ &= \frac{1}{4}\int [1-\cos^2(2u)]\,du.\end{aligned}$$

Again we apply the half-angle formula:

$$\begin{aligned}\int e^x \sin^2(e^x)\cos^2(e^x)\,dx &= \frac{1}{4}\int \left[1 - \frac{1+\cos(4u)}{2}\right]du \\ &= \frac{1}{4}\int \left[\frac{1}{2} - \frac{1}{2}\cos(4u)\right]du \\ &= \frac{1}{8}\int [1-\cos(4u)]\,du \\ &= \frac{1}{8}\left[u - \frac{1}{4}\sin(4u)\right] + C \\ &= \frac{1}{8}e^x - \frac{1}{32}\sin(4e^x) + C.\end{aligned}$$