MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 4 | **MATHEMATICS 1001** | FALL 2019

SOLUTIONS

[3] 1. (a) We can write

$$\sum_{i=1}^{n} i(i+1) = 2 + 6 + 12 + 20 + 30 + \cdots + n(n+1).$$

Furthermore,

$$\begin{aligned}\sum_{i=1}^{n} i(i+1) &= \sum_{i=1}^{n} (i^2 + i) \\ &= \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i \\ &= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \\ &= \frac{1}{3}n^3 + n^2 + \frac{2}{3}n.\end{aligned}$$

[3] (b) We can write

$$\sum_{i=1}^{n} (i^3 - 8i + 3) = -4 - 5 + 6 + 35 + 88 + \cdots + (n^3 - 8n + 3).$$

We have

$$\begin{aligned}\sum_{i=1}^{n} (i^3 - 5i + 3) &= \sum_{i=1}^{n} i^3 - 8\sum_{i=1}^{n} i + 3\sum_{i=1}^{n} 1 \\ &= \frac{n^2(n+1)^2}{4} - 8 \cdot \frac{n(n+1)}{2} - 3 \cdot n \\ &= \frac{1}{4}n^4 + \frac{1}{2}n^3 - \frac{15}{4}n^2 - n.\end{aligned}$$

[1] 2. (a) Since $a = 2$ and $b = 4$,

$$\Delta x = \frac{b-a}{n} = \frac{2}{n}.$$

[2] (b) We have

$$x_i = a + i\Delta x = 2 + \frac{2i}{n}.$$

This function is decreasing on $[2, 4]$ (as can be seen by sketching the graph or via calculus) so the minimum value of $f(x)$ on each subinterval occurs at the right endpoint, while the maximum value occurs at the left endpoint. Hence

$$m_i = x_i = 2 + \frac{2i}{n} \quad \text{and} \quad M_i = x_{i-1} = 2 + \frac{2(i-1)}{n}.$$

[2] (c) First,

$$
\begin{aligned}
f(m_i) &= f\left(2 + \frac{2i}{n}\right) \\
&= 4\left(2 + \frac{2i}{n}\right) - \left(2 + \frac{2i}{n}\right)^2 \\
&= 4 - \frac{4i^2}{n^2}.
\end{aligned}
$$

Next,

$$
\begin{aligned}
f(M_i) &= f\left(2 + \frac{2(i-1)}{n}\right) \\
&= 4\left(2 + \frac{2(i-1)}{n}\right) - \left(2 + \frac{2(i-1)}{n}\right)^2 \\
&= 4 - \frac{4i^2}{n^2} + \frac{8i}{n^2} - \frac{4}{n^2}.
\end{aligned}
$$

[4] (d) The lower sum will be

$$
\begin{aligned}
s(n) &= \sum_{i=1}^{n} f(m_i)\Delta x \\
&= \sum_{i=1}^{n} \left[4 - \frac{4i^2}{n^2}\right] \cdot \frac{2}{n} \\
&= \frac{8}{n}\sum_{i=1}^{n} 1 - \frac{8}{n^3}\sum_{i=1}^{n} i^2 \\
&= \frac{8}{n} \cdot n - \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \\
&= \frac{16}{3} - \frac{4}{n} - \frac{4}{3n^2}.
\end{aligned}
$$

For the upper sum, we have

$$\begin{aligned} S(n) &= \sum_{i=1}^{n} f(M_i)\Delta x \\ &= \sum_{i=1}^{n} \left[4 - \frac{4i^2}{n^2} + \frac{8i}{n^2} - \frac{4}{n^2}\right] \cdot \frac{2}{n} \\ &= \frac{8}{n}\sum_{i=1}^{n} 1 - \frac{8}{n^3}\sum_{i=1}^{n} i^2 + \frac{16}{n^3}\sum_{i=1}^{n} i - \frac{8}{n^3}\sum_{i=1}^{n} 1 \\ &= \frac{8}{n} \cdot n - \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{16}{n^3} \cdot \frac{n(n+1)}{2} - \frac{8}{n^3} \cdot n \\ &= \frac{16}{3} + \frac{4}{n} - \frac{4}{3n^2}. \end{aligned}$$

[2] (e) We have

$$s(5) = \frac{16}{3} - \frac{4}{5} - \frac{4}{3 \cdot 25} = \frac{112}{25} = 4.48$$

and

$$S(5) = \frac{16}{3} + \frac{4}{5} - \frac{4}{3 \cdot 25} = \frac{152}{25} = 6.08.$$

[2] (f) We have

$$s(500) = \frac{16}{3} - \frac{4}{500} - \frac{4}{3 \cdot 250000} = \frac{328333}{62500} \approx 5.25$$

and

$$S(500) = \frac{16}{3} + \frac{4}{500} - \frac{4}{3 \cdot 250000} = \frac{338333}{62500} \approx 5.41.$$

[2] (g) Observe that

$$\lim_{n\to\infty} s(n) = \lim_{n\to\infty} \left[\frac{16}{3} - \frac{4}{n} - \frac{4}{3n^2}\right] = \frac{16}{3}$$

and

$$\lim_{n\to\infty} S(n) = \lim_{n\to\infty} \left[\frac{16}{3} + \frac{4}{n} - \frac{4}{3n^2}\right] = \frac{16}{3},$$

so $A = \frac{16}{3} \approx 5.33$.

[7] 3. (a) We form a regular partition where

$$\Delta x = \frac{2-(-1)}{n} = \frac{3}{n}$$

and choose the sample point

$$x_i^* = x_i = -1 + \frac{3i}{n}.$$

Thus

$$\begin{aligned} f(x_i) &= f\left(-1+\frac{3i}{n}\right) \\ &= \left(-1+\frac{3i}{n}\right)^3+5 \\ &= \frac{27i^3}{n^3}-\frac{27i^2}{n^2}+\frac{9i}{n}+4. \end{aligned}$$

Now we have

$$\begin{aligned} A &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x \\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{27i^3}{n^3}-\frac{27i^2}{n^2}+\frac{9i}{n}+4\right]\cdot\frac{3}{n} \\ &= \lim_{n\to\infty}\left[\frac{81}{n^4}\sum_{i=1}^{n} i^3-\frac{81}{n^3}\sum_{i=1}^{n} i^2+\frac{27}{n^2}\sum_{i=1}^{n} i+\frac{12}{n}\sum_{i=1}^{n} 1\right] \\ &= \lim_{n\to\infty}\left[\frac{81}{n^4}\cdot\frac{n^2(n+1)^2}{4}-\frac{81}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}+\frac{27}{n^2}\cdot\frac{n(n+1)}{2}+\frac{12}{n}\cdot n\right] \\ &= \frac{81}{4}-27+\frac{27}{2}+12 \\ &= \frac{75}{4}. \end{aligned}$$

[7] (b) We form a regular partition where

$$\Delta x = \frac{5-0}{n} = \frac{5}{n}$$

and choose the sample point

$$x_i^* = x_i = 0+\frac{5i}{n} = \frac{5i}{n}.$$

Thus

$$\begin{aligned} f(x_i) &= f\left(\frac{5i}{n}\right) \\ &= \left(3\cdot\frac{5i}{n}-1\right)^2 \\ &= \frac{225i^2}{n^2}-\frac{30i}{n}+1. \end{aligned}$$

Now we have

$$
\begin{aligned}
A &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{225i^2}{n^2} - \frac{30i}{n} + 1\right] \cdot \frac{5}{n} \\
&= \lim_{n\to\infty} \left[\frac{1125}{n^3}\sum_{i=1}^{n} i^2 - \frac{150}{n^2}\sum_{i=1}^{n} i + \frac{5}{n}\sum_{i=1}^{n} 1\right] \\
&= \lim_{n\to\infty} \left[\frac{1125}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{150}{n^2} \cdot \frac{n(n+1)}{2} + \frac{5}{n} \cdot n\right] \\
&= 375 - 75 + 5 \\
&= 305.
\end{aligned}
$$

[5] 4. Since the y-axis is the line $x = 0$, this region lies on the interval $[0, b]$. Thus we form a regular partition where

$$\Delta x = \frac{b-0}{n} = \frac{b}{n}$$

and choose the sample point

$$x_i^* = x_i = 0 + \frac{bi}{n} = \frac{bi}{n}.$$

So then

$$f(x_i) = f\left(\frac{bi}{n}\right) = \left(\frac{bi}{n}\right)^2 = \frac{b^2 i^2}{n^2}.$$

Now we have

$$
\begin{aligned}
A &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{b^2 i^2}{n^2} \cdot \frac{b}{n} \\
&= \lim_{n\to\infty} \left[\frac{b^3}{n^3} \sum_{i=1}^{n} i^2\right] \\
&= \lim_{n\to\infty} \left[\frac{b^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}\right] \\
&= b^3 \cdot \frac{1}{3} \\
&= \frac{b^3}{3}
\end{aligned}
$$

as desired.