

MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 4.5

Math 1001 Worksheet

FALL 2025

SOLUTIONS

1. (a) Observe that $f(x) < 0$ when $0 \leq x < 1$. Hence it is not a non-negative function for all x , and therefore it is not a probability density function.
- (b) We can see that $f(x) \geq 0$ for all x , so we evaluate

$$\begin{aligned}\int_{-\infty}^{\infty} f(x) dx &= \int_0^8 \frac{1}{(x+4)^3} dx \\ &= \left[-\frac{1}{2(x+4)^2} \right]_0^8 \\ &= -\frac{1}{288} + \frac{1}{32} \\ &= \frac{1}{36}.\end{aligned}$$

Since $\int_{-\infty}^{\infty} f(x) dx \neq 1$, this is not a probability density function.

- (c) Again we can see that $f(x) \geq 0$ for all x , so we evaluate

$$\int_{-\infty}^{\infty} f(x) dx = \int_0^8 \frac{18x}{(x+4)^3} dx.$$

We let $u = x + 4$ so $du = dx$ and $x = u - 4$. When $x = 0$, $u = 4$. When $x = 8$, $u = 12$. The integral becomes

$$\begin{aligned}\int_{-\infty}^{\infty} f(x) dx &= 18 \int_4^{12} \frac{u-4}{u^3} du \\ &= 18 \int_4^{12} \left(\frac{1}{u^2} - \frac{4}{u^3} \right) du \\ &= 18 \left[-\frac{1}{u} + \frac{2}{u^2} \right]_4^{12} \\ &= 18 \left[-\frac{1}{12} + \frac{1}{72} + \frac{1}{4} - \frac{1}{8} \right] \\ &= 1.\end{aligned}$$

Hence this is a probability density function.

2. (a) Note that $f(x) \geq 0$ for all x as long as $k \geq 0$, and that

$$\begin{aligned}
 \int_{-\infty}^{\infty} f(x) dx &= \int_{-\infty}^{\infty} \frac{k}{x^2 + 1} dx \\
 &= \int_{-\infty}^0 \frac{k}{x^2 + 1} dx + \int_0^{\infty} \frac{k}{x^2 + 1} dx \\
 &= k \lim_{T \rightarrow -\infty} \int_T^0 \frac{1}{x^2 + 1} dx + k \lim_{S \rightarrow \infty} \int_0^S \frac{1}{x^2 + 1} dx \\
 &= k \lim_{T \rightarrow -\infty} \left[\arctan(x) \right]_T^0 + k \lim_{S \rightarrow \infty} \left[\arctan(x) \right]_0^S \\
 &= k \lim_{T \rightarrow -\infty} [0 - \arctan(T)] + k \lim_{S \rightarrow \infty} [\arctan(S) - 0] \\
 &= k \cdot \frac{\pi}{2} + k \cdot \frac{\pi}{2} \\
 &= k\pi.
 \end{aligned}$$

Thus we must have $k\pi = 1$ and so $k = \frac{1}{\pi}$.

(b) We have

$$\begin{aligned}
 P(-\sqrt{3} \leq X \leq \sqrt{3}) &= \int_{-\sqrt{3}}^{\sqrt{3}} f(x) dx \\
 &= \frac{1}{\pi} \int_{-\sqrt{3}}^{\sqrt{3}} \frac{1}{x^2 + 1} dx \\
 &= \frac{1}{\pi} \left[\arctan(x) \right]_{-\sqrt{3}}^{\sqrt{3}} \\
 &= \frac{1}{\pi} \left[\arctan(\sqrt{3}) - \arctan(-\sqrt{3}) \right] \\
 &= \frac{1}{\pi} \left[\frac{\pi}{3} + \frac{\pi}{3} \right] \\
 &= \frac{2}{3}.
 \end{aligned}$$

3. (a) Note that $f(x) \geq 0$ for all x as long as $k \geq 0$, and that

$$\begin{aligned}
 \int_{-\infty}^{\infty} f(x) dx &= k \int_0^{\infty} x e^{-2x} dx \\
 &= k \lim_{T \rightarrow \infty} \int_0^T x e^{-2x} dx.
 \end{aligned}$$

We use integration by parts with $w = x$ so $dw = dx$, and $dv = e^{-2x} dx$ so $v = -\frac{1}{2}e^{-2x}$.

Then

$$\begin{aligned}\int_{-\infty}^{\infty} f(x) dx &= k \lim_{T \rightarrow \infty} \left(\left[-\frac{1}{2} x e^{-2x} \right]_0^T + \frac{1}{2} \int_0^T e^{-2x} dx \right) \\&= k \lim_{T \rightarrow \infty} \left[-\frac{1}{2} x e^{-2x} - \frac{1}{4} e^{-2x} \right]_0^T \\&= k \lim_{T \rightarrow \infty} \left[-\frac{1}{2} T e^{-2T} - \frac{1}{4} e^{-2T} + 0 + \frac{1}{4} \right] \\&= \frac{1}{4} k - \frac{1}{2} k \lim_{T \rightarrow \infty} \frac{T}{e^{2T}} \\&\stackrel{\text{H}}{=} \frac{1}{4} k - \frac{1}{2} k \lim_{T \rightarrow \infty} \frac{1}{2e^{2T}} \\&= \frac{1}{4} k - 0 \\&= \frac{1}{4} k.\end{aligned}$$

Hence we must have $\frac{1}{4}k = 1$ and so $k = 4$.

(b) We have

$$\begin{aligned}P(0 \leq X \leq 2) &= \int_0^1 f(x) dx \\&= 4 \int_0^2 x e^{-2x} dx \\&= 4 \left[-\frac{1}{2} x e^{-2x} - \frac{1}{4} e^{-2x} \right]_0^2 \\&= 4 \left[-e^{-4} - \frac{1}{4} e^{-4} + \frac{1}{4} \right] \\&= 1 - 5e^{-4} \\&\approx 0.9.\end{aligned}$$

4. (a) Using the same substitution as in Question 1(c), this probability is given by

$$\begin{aligned}
 P(3 \leq X \leq 5) &= \int_3^5 f(x) \, dx \\
 &= 18 \int_3^5 \frac{x}{(x+4)^3} \, dx \\
 &= 18 \int_7^9 \left(\frac{1}{u^2} - \frac{4}{u^3} \right) \, du \\
 &= 18 \left[-\frac{1}{u} + \frac{2}{u^2} \right]_7^9 \\
 &= 18 \left[-\frac{1}{9} + \frac{2}{81} + \frac{1}{7} - \frac{2}{49} \right] \\
 &= \frac{124}{441}.
 \end{aligned}$$

Hence the likelihood is about 28.1% that the total viewing time will be between 3 minutes and 5 minutes.

- (b) Using the same substitution as in Question 1(c), this probability is given by

$$\begin{aligned}
 P(0 \leq X \leq 1) &= \int_0^1 f(x) \, dx \\
 &= 18 \int_0^1 \frac{x}{(x+4)^3} \, dx \\
 &= 18 \int_4^5 \left(\frac{1}{u^2} - \frac{4}{u^3} \right) \, du \\
 &= 18 \left[-\frac{1}{u} + \frac{2}{u^2} \right]_4^5 \\
 &= 18 \left[-\frac{1}{5} + \frac{2}{25} + \frac{1}{4} - \frac{1}{8} \right] \\
 &= \frac{9}{100}.
 \end{aligned}$$

Hence there's a 9% likelihood that the total viewing time will be less than 1 minute.

(c) Using the same substitution as in Question 1(c), this probability is given by

$$\begin{aligned}
 P(6 \leq X \leq 8) &= \int_6^8 f(x) \, dx \\
 &= 18 \int_6^8 \frac{x}{(x+4)^3} \\
 &\quad , \, dx \\
 &= 18 \int_{10}^{12} \left(\frac{1}{u^2} - \frac{4}{u^3} \right) \, du \\
 &= 18 \left[-\frac{1}{u} + \frac{2}{u^2} \right]_{10}^{12} \\
 &= 18 \left[-\frac{1}{12} + \frac{1}{72} + \frac{1}{10} - \frac{1}{50} \right] \\
 &= \frac{19}{100}.
 \end{aligned}$$

Hence there's a 19% likelihood that the total viewing time will be greater than 6 minutes.

(d) We have

$$\begin{aligned}
 \mu &= \int_{-\infty}^{\infty} f(x) \, dx \\
 &= \int_0^8 x \cdot \frac{18x}{(x+4)^3} \, dx \\
 &= 18 \int_0^8 \frac{x^2}{(x+4)^3} \, dx.
 \end{aligned}$$

Let $u = x + 4$ so $du = dx$ and $x^2 = (u - 4)^2$. When $x = 0$, $u = 4$. When $x = 8$, $u = 12$.

The integral becomes

$$\begin{aligned}\mu &= 18 \int_4^{12} \frac{(u-4)^2}{u^3} du \\&= 18 \int_4^{12} \frac{u^2 - 8u + 16}{u^3} du \\&= 18 \int_4^{12} \left(\frac{1}{u} - \frac{8}{u^2} + \frac{16}{u^3} \right) du \\&= 18 \left[\ln|u| + \frac{8}{u} - \frac{8}{u^2} \right]_4^{12} \\&= 18 \left[\ln(12) + \frac{2}{3} - \frac{1}{18} - \ln(4) - 2 + \frac{1}{2} \right] \\&= 18 \ln(12) - 18 \ln(4) - 16 \\&\approx 3.8.\end{aligned}$$

5. (a) Clearly, $f(x) \geq 0$ for all x , and furthermore

$$\begin{aligned}\int_{-\infty}^{\infty} f(x) dx &= \int_0^5 \frac{1}{5} dx \\&= \frac{1}{5} [x]_0^5 \\&= \frac{1}{5} \cdot 5 \\&= 1.\end{aligned}$$

Hence $f(x)$ is a probability density function.

Next,

$$\begin{aligned}\mu &= \int_{-\infty}^{\infty} x f(x) dx \\&= \int_0^5 \frac{1}{5} x dx \\&= \frac{1}{5} \left[\frac{1}{2} x^2 \right]_0^5 \\&= \frac{1}{5} \left[\frac{25}{2} - 0 \right] \\&= \frac{5}{2}.\end{aligned}$$

Note that this makes sense, since it's the midpoint of the interval $[0, 5]$.

(b) We want a density function such that, for some constant k ,

$$f(x) = \begin{cases} k, & \text{for } 0 \leq x < 4 \\ 2k, & \text{for } 4 \leq x \leq 5 \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, $f(x) \geq 0$ for all x as long $k \geq 0$. Then we have

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) dx &= \int_0^4 k dx + \int_4^5 2k dx \\ &= k \left[x \right]_0^4 + 2k \left[x \right]_4^5 \\ &= k(4 - 0) + 2k(5 - 4) \\ &= 6k, \end{aligned}$$

so we need $6k = 1$ therefore $k = \frac{1}{6}$. Hence a suitable probability density function is given by

$$f(x) = \begin{cases} \frac{1}{6}, & \text{for } 0 \leq x < 4 \\ \frac{1}{3}, & \text{for } 4 \leq x \leq 5 \\ 0, & \text{otherwise.} \end{cases}$$

Finally,

$$\begin{aligned} \mu &= \int_{-\infty}^{\infty} x f(x) dx \\ &= \frac{1}{6} \int_0^4 x dx + \frac{1}{3} \int_4^5 x dx \\ &= \frac{1}{6} \left[\frac{1}{2} x^2 \right]_0^4 + \frac{1}{3} \left[\frac{1}{2} x^2 \right]_4^5 \\ &= \frac{1}{6} [8 - 0] + \frac{1}{3} \left[\frac{25}{2} - 8 \right] \\ &= \frac{17}{6}. \end{aligned}$$

In other words, the effect of Ruby's error is to shift the mean value of the probability distribution from 2.5 to $2.8\bar{3}$.