MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS



## SOLUTIONS

1. If we substitute $t = 0$ and $y = y_0$ into the implicit solution, we have

$$\frac{y_0}{1 - my_0} = Ce^0 \quad \Longrightarrow \quad C = \frac{y_0}{1 - my_0}.$$

2. (a) We already know that the solution of the logistic differential equation has the form

$$y = \frac{Ce^{kt}}{1 + Cme^{kt}}.$$

From Question 1, we also have

$$C = \frac{y_0}{1 - my_0}.$$

Since $y_0 = 50$, $k = \frac{1}{5}$ and $m = \frac{1}{1000}$, we have

$$C = \frac{50}{1 - \frac{1}{1000} \cdot 50} = \frac{1000}{19}$$

and so

$$y = \frac{\frac{1000}{19} e^{\frac{1}{5} \cdot t}}{1 + \frac{1000}{19} \cdot \frac{1}{1000} \cdot e^{\frac{t}{5}}} = \frac{\frac{1000}{19} e^{\frac{1}{5} \cdot t}}{1 + \frac{1}{19} e^{\frac{t}{5}}} = \frac{1000e^{\frac{t}{5}}}{19 + e^{\frac{t}{5}}}.$$

Hence

$$y(5) = \frac{1000e}{19 + e} \approx 125.$$

After 5 years, there are approximately 125 rabbits in the colony.

(b) We set

$$\begin{aligned} \frac{1000e^{\frac{t}{5}}}{19 + e^{\frac{t}{5}}} &= 500 \\ 1000e^{\frac{t}{5}} &= 9500 + 500e^{\frac{t}{5}} \\ 500e^{\frac{t}{5}} &= 9500 \\ e^{\frac{t}{5}} &= 19 \\ \frac{t}{5} &= \ln(19) \\ t &= 5\ln(19) \approx 15. \end{aligned}$$

It takes about 15 years for the colony to number 500 rabbits.

(c) Since the logistic differential equation has the form

$$\frac{dy}{dt} = ky(1 - my),$$

where $\frac{1}{m}$ is the carrying capacity, we have

$$\frac{1}{m} = \frac{1}{\frac{1}{1000}} = 1000.$$

In other words, the population will eventually approach 1000 rabbits.

3. While we could make use of the solution of the logistic differential equation, there is no need to do so. All we have to do is note that the carrying capacity of both populations is the same: $m = \frac{1}{800}$ so the carrying capacity is

$$\frac{1}{m} = \frac{1}{\frac{1}{800}} = 800.$$

This tells us that both populations will approach a size of 800 fruit flies as $t \to \infty$. The first population will grow to reach its carrying capacity, while the second population will contract.

4. (a) Under the natural growth model, the population of Barovia would be described by

$$y = y_0 e^{kt}$$

where $y_0 = 50$ and $y(10) = 51200$. Thus

$$51200 = 50e^{10k} \quad \Longrightarrow \quad 1024 = e^{10k} \quad \Longrightarrow \quad k = \frac{1}{10}\ln(1024).$$

We know that constant harvesting can be described by the differential equation

$$\frac{dy}{dt} = ky - h,$$

where $h = 26$ in this case. We have found the solution to this differential equation to be

$$y = y_0 e^{kt} + H(1 - e^{kt})$$

where

$$H = \frac{h}{k} = \frac{26}{\frac{1}{10}\ln(1024)} = \frac{260}{\ln(1024)}.$$

Thus we have

$$y = 50e^{\frac{1}{10}\ln(1024)t} + \frac{260}{\ln(1024)}(1 - e^{\frac{1}{10}\ln(1024)t})$$

and so

$$\begin{aligned} y(10) &= 50e^{\ln(1024)} + \frac{260}{\ln(1024)}(1 - e^{\ln(1024)}) \\ &= 50 \cdot 1024 + \frac{260}{\ln(1024)}(1 - 1024) \\ &= 51200 - \frac{265980}{\ln(1024)} \\ &\approx 12827. \end{aligned}$$

Thus the population of Barovia after 10 years is actually just $12,827$ villagers.

(b) We have seen that a population will survive as long as $H < y_0$, so we must have

$$\frac{h}{\frac{1}{10}\ln(1024)} < 50 \quad \Longrightarrow \quad h < 5\ln(1024).$$

Since $5\ln(1024) \approx 34.7$, the population will tend towards extinction if Count Strahd takes away a villager every week — that is, when $h = 52$.

5. (a) Since effort harvesting will remove an amount $hy$ from the population, a suitable differential equation is

$$\frac{dy}{dt} = ky - hy.$$

(b) This is a separable differential equation, because we can write

$$\begin{aligned} \frac{dy}{dt} &= (k-h)y \\ \frac{1}{y}\,dy &= (k-h)\,dt \\ \int \frac{1}{y}\,dy &= (k-h)\int dt \\ \ln|y| &= (k-h)t + C \\ y &= e^{(k-h)t+C} \\ &= Ce^{(k-h)t}, \end{aligned}$$

where we have assumed $y > 0$ so that we can neglect the absolute value in the logarithm. Since $y(0) = y_0$, we have

$$y(0) = Ce^{(k-h)\cdot 0} = C = y_0,$$

so the particular solution is

$$y = y_0 e^{(k-h)t}.$$

(c) From the form of the solution, we can see that

$$\lim_{t\to\infty} y = \lim_{t\to\infty} y_0 e^{(k-h)t} = 0$$

if $(k-h) < 0$, which means that $k < h$. Thus the population will survive under effort harvesting only if $k \geq h$.