MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

AREA UNDER A CURVE **Math 1001 Worksheet** FALL 2019

SOLUTIONS

1. (a) From the nth term, we see that this sum can be written

$$\sum_{i=1}^{n} \frac{2^i}{5i}.$$

(b) The form of the ith term is clearly indicated by the last term in the sum, so we have

$$\sum_{i=1}^{n} (i-2)^3 y^2.$$

2. (a) $\displaystyle\sum_{i=1}^{n}(4i+3) = 4\sum_{i=1}^{n} i + 3\sum_{i=1}^{n} 1 = 4\left(\frac{n(n+1)}{2}\right) + 3n = 2n^2 + 5n$

(b) $$\sum_{i=1}^{n}(i^3-6i) = \sum_{i=1}^{n} i^3 - 6\sum_{i=1}^{n} i = \frac{n^2(n+1)^2}{4} - 6\left(\frac{n(n+1)}{2}\right)$$
$$= \frac{1}{4}n^4 + \frac{1}{2}n^3 - \frac{11}{4}n^2 - 3n$$

(c) $$\sum_{i=1}^{n}(3i+1)^2 = \sum_{i=1}^{n}(9i^2+6i+1) = 9\sum_{i=1}^{n} i^2 + 6\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$$
$$= 9\left(\frac{n(n+1)(2n+1)}{6}\right) + 6\left(\frac{n(n+1)}{2}\right) + n = 3n^3 + \frac{21}{2}n^2 + \frac{17}{2}n$$

3. The graph of R can be found in Figure 1.

(a) For a regular partition of $[-1, 1]$ into n subintervals,

$$\Delta x = \frac{1-(-1)}{n} = \frac{2}{n}.$$

(b) First we have

$$x_i = -1 + \frac{2i}{n}.$$

Next observe that $f(x)$ is decreasing on the interval $[-1, 1]$, so for any partition, $m_i = x_i$ (the right endpoint) and $M_i = x_{i-1}$ (the left endpoint). Thus

$$m_i = -1 + \frac{2i}{n} \quad \text{and} \quad M_i = -1 + \frac{2(i-1)}{n}.$$

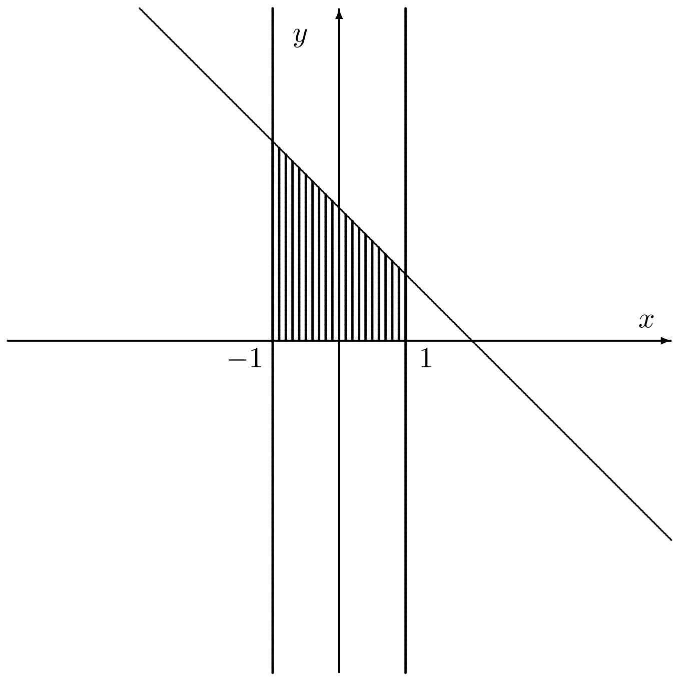

Figure 1: The region under the curve $f(x) = 2 - x$ on the interval $[-1, 1]$, as considered in Question 3.

(c) We have

$$f(m_i) = f\left(-1 + \frac{2i}{n}\right) = 2 - \left(-1 + \frac{2i}{n}\right) = 3 - \frac{2i}{n}$$

and

$$f(M_i) = f\left(-1 + \frac{2(i-1)}{n}\right) = 2 - \left(-1 + \frac{2(i-1)}{n}\right) = 3 - \frac{2i}{n} + \frac{2}{n}.$$

(d) The lower sum is

$$\begin{aligned} S(n) &= \sum_{i=1}^{n} f(m_i)\Delta x \\ &= \sum_{i=1}^{n} \left(3 - \frac{2i}{n}\right) \cdot \frac{2}{n} \\ &= \frac{6}{n}\sum_{i=1}^{n} 1 - \frac{4}{n^2}\sum_{i=1}^{n} i \\ &= \frac{6}{n} \cdot n - \frac{4}{n^2} \cdot \frac{n(n+1)}{2} \\ &= 6 - \frac{2(n+1)}{n} \\ &= 4 - \frac{2}{n}. \end{aligned}$$

The upper sum is

$$\begin{aligned} s(n) &= \sum_{i=1}^{n} f(M_i)\Delta x \\ &= \sum_{i=1}^{n} \left(3 - \frac{2i}{n} + \frac{2}{n}\right) \cdot \frac{2}{n} \\ &= \frac{6}{n}\sum_{i=1}^{n} 1 - \frac{4}{n^2}\sum_{i=1}^{n} i + \frac{4}{n^2}\sum_{i=1}^{n} 1 \\ &= \frac{6}{n}\cdot n - \frac{4}{n^2}\cdot\frac{n(n+1)}{2} + \frac{4}{n^2}\cdot n \\ &= 6 - \frac{2(n+1)}{n} + \frac{4}{n} \\ &= 4 + \frac{2}{n}. \end{aligned}$$

(e) We have

$$s(50) = 4 - \frac{2}{5} = \frac{18}{5} = 3.6$$

and

$$S(5) = 4 + \frac{2}{5} = \frac{22}{5} = 4.4.$$

(f) We have

$$s(500) = 4 - \frac{2}{500} = \frac{999}{250} = 3.996$$

and

$$S(5) = 4 + \frac{2}{500} = \frac{1001}{250} = 4.004.$$

(g) We have

$$\lim_{n\to\infty} s(n) = \lim_{n\to\infty}\left(4 - \frac{2}{n}\right) = 4 - 0 = 4$$

and

$$\lim_{n\to\infty} S(n) = \lim_{n\to\infty}\left(4 + \frac{2}{n}\right) = 4 + 0 = 4.$$

Hence $A = 4$.

4. We will use a regular partition where

$$\Delta x = \frac{3-(-1)}{n} = \frac{4}{n}$$

and choose as our sample point

$$x_i^* = x_i = -1 + \frac{4i}{n}.$$

Thus

$$f(x_i^*) = 3 + 3\left(-1 + \frac{4i}{n}\right) - \left(-1 + \frac{4i}{n}\right)^2 = -\frac{16}{n^2}i^2 + \frac{20}{n}i - 1.$$

The area A of the region is given by

$$\begin{aligned}
A &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(-\frac{16}{n^2}i^2 + \frac{20}{n}i - 1\right) \cdot \frac{4}{n} \\
&= \lim_{n\to\infty} \left(-\frac{64}{n^3} \sum_{i=1}^{n} i^2 + \frac{80}{n^2} \sum_{i=1}^{n} i - \frac{4}{n} \sum_{i=1}^{n} 1\right) \\
&= \lim_{n\to\infty} \left(-\frac{64}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{80}{n^2} \cdot \frac{n(n+1)}{2} - \frac{4}{n} \cdot n\right) \\
&= \lim_{n\to\infty} \left(-\frac{32(n+1)(2n+1)}{3n^2} + \frac{40(n+1)}{n} - 4\right) \\
&= -\frac{64}{3} + 40 - 4 \\
&= \frac{44}{3}.
\end{aligned}$$