MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 3 | **MATHEMATICS 1001** | FALL 2019

SOLUTIONS

[5] 1. (a) First observe that

$$\int \frac{x^3}{e^{3x}}\,dx = \int x^3 e^{-3x}\,dx.$$

Now let $w = x^3$ so $dw = 3x^2\,dx$, and let $dv = e^{-3x}\,dx$ so $v = -\frac{1}{3}e^{-3x}$. Using the integration by parts formula, we have

$$\begin{aligned}\int \frac{x^3}{e^{3x}}\,dx &= -\frac{1}{3}x^3e^{-3x} - \int\left(-\frac{1}{3}e^{-3x}\right)\cdot 3x^2\,dx \\ &= -\frac{1}{3}x^3e^{-3x} + \int x^2e^{-3ex}\,dx.\end{aligned}$$

We need integration by parts again for the new integral. We let $w = x^2$ so $dw = 2x\,dx$, and again $dv = e^{-3x}\,dx$ so $v = -\frac{1}{3}e^{-3x}$. Then

$$\begin{aligned}\int x^2e^{-3ex}\,dx &= -\frac{1}{3}x^2e^{-3x} - \int\left(-\frac{1}{3}e^{-3x}\right)\cdot 2x\,dx \\ &= -\frac{1}{3}x^2e^{-3x} + \frac{2}{3}\int xe^{-3ex}\,dx\end{aligned}$$

and therefore

$$\int \frac{x^3}{e^{3x}}\,dx = -\frac{1}{3}x^3e^{-3x} - \frac{1}{3}x^2e^{-3x} + \frac{2}{3}\int xe^{-3ex}\,dx.$$

We need integration by parts one more time. We let $w = x$ so $dw = dx$, and again $dv = e^{-3x}\,dx$ so $v = -\frac{1}{3}e^{-3x}$. Then

$$\begin{aligned}\int xe^{-3ex}\,dx &= -\frac{1}{3}xe^{-3ex} + \frac{1}{3}\int e^{-3ex}\,dx \\ &= -\frac{1}{3}xe^{-3ex} + \frac{1}{3}\left(-\frac{1}{3}e^{-3ex}\right) + C \\ &= -\frac{1}{3}xe^{-3ex} - \frac{1}{9}e^{-3ex} + C,\end{aligned}$$

and so

$$\begin{aligned}\int \frac{x^3}{e^{3x}}\,dx &= -\frac{1}{3}x^3e^{-3x} - \frac{1}{3}x^2e^{-3x} + \frac{2}{3}\left(-\frac{1}{3}xe^{-3ex} - \frac{1}{9}e^{-3ex}\right) + C \\ &= -\frac{1}{3}x^3e^{-3x} - \frac{1}{3}x^2e^{-3x} - \frac{2}{9}xe^{-3ex} - \frac{2}{27}e^{-3x} + C.\end{aligned}$$

[4] (b) Let $w = \ln^2(x)$ so $dw = \dfrac{2\ln(x)}{x}\,dx$. Let $dv = dx$ so $v = x$. Then

$$\begin{aligned}\int \ln^2(x)\,dx &= x\ln^2(x) - \int x \cdot \frac{2\ln(x)}{x}\,dx \\ &= x\ln^2(x) - 2\int \ln(x)\,dx.\end{aligned}$$

Here we can use integration by parts again (or our result from class) to find that

$$\int \ln(x)\,dx = x\ln(x) - x + C,$$

and so

$$\begin{aligned}\int \ln^2(x)\,dx &= x\ln^2(x) - 2[x\ln(x) - x] + C \\ &= x\ln^2(x) - 2x\ln(x) + 2x + C.\end{aligned}$$

[5] (c) Let $w = \arctan(x)$ so $dw = \dfrac{1}{1+x^2}\,dx$. Let $dv = x\,dx$ so $v = \frac{1}{2}x^2$. Then

$$\int x\arctan(x)\,dx = \frac{1}{2}x^2\arctan(x) - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx.$$

To evaluate the remaining integral, in principal we need long division (because this is an improper rational function). However, because the numerator and denominator are so similar, we can more straightforwardly write:

$$\begin{aligned}\int \frac{x^2}{1+x^2}\,dx &= \int \frac{1+x^2-1}{1+x^2}\,dx \\ &= \int \left(\frac{1+x^2}{1+x^2} - \frac{1}{1+x^2}\right)\,dx \\ &= \int \left(1 - \frac{1}{1+x^2}\right)\,dx \\ &= x - \arctan(x) + C.\end{aligned}$$

Either way, we now have

$$\begin{aligned}\int x\arctan(x)\,dx &= \frac{1}{2}x^2\arctan(x) - \frac{1}{2}[x - \arctan(x)] + C \\ &= \frac{1}{2}x^2\arctan(x) - \frac{1}{2}x + \frac{1}{2}\arctan(x) + C.\end{aligned}$$

[4] (d) Let $w = \sinh(3t)$ so $dw = 3\cosh(3t)\,dt$. Let $dv = \cos(7t)\,dt$ so $v = \frac{1}{7}\sin(7t)$. Then

$$\int \sinh(3t)\cos(7t)\,dt = \frac{1}{7}\sinh(3t)\sin(7t) - \frac{3}{7}\int \cosh(3t)\sin(7t)\,dt.$$

Now we use integration by parts again. Let $w = \cosh(3t)$ so $dw = 3\sinh(3t)$. Let $dv = \sin(7t)\,dt$ so $v = -\frac{1}{7}\cos(7t)$. We have

$$\int \cosh(3t)\sin(7t)\,dt = -\frac{1}{7}\cosh(3t)\cos(7t) + \frac{3}{7}\int \sinh(3t)\cos(7t)\,dt.$$

Now we note that the original integral has reoccurred, so putting all of this together we have

$$\begin{aligned}
\int \sinh(3t)\cos(7t)\,dt &= \frac{1}{7}\sinh(3t)\sin(7t) \\
&\quad - \frac{3}{7}\left[-\frac{1}{7}\cosh(3t)\cos(7t) + \frac{3}{7}\int \sinh(3t)\cos(7t)\,dt\right] \\
&= \frac{1}{7}\sinh(3t)\sin(7t) + \frac{3}{49}\cosh(3t)\cos(7t) - \frac{9}{49}\int \sinh(3t)\cos(7t)\,dt \\
\frac{58}{49}\int \sinh(3t)\cos(7t)\,dt &= \frac{1}{7}\sinh(3t)\sin(7t) + \frac{3}{49}\cosh(3t)\cos(7t) + C \\
\int \sinh(3t)\cos(7t)\,dt &= \frac{7}{58}\sinh(3t)\sin(7t) + \frac{3}{58}\cosh(3t)\cos(7t) + C.
\end{aligned}$$

[4] 2. (a) Let $w = \sec^{n-2}(x)$ so

$$dw = (n-2)\sec^{n-3}(x)\cdot\sec(x)\tan(x)\,dx = (n-2)\sec^{n-2}(x)\tan(x)\,dx.$$

Let $dv = \sec^2(x)\,dx$ so $v = \tan(x)$. Then

$$\int \sec^n(x)\,dx = \tan(x)\sec^{n-2}(x) - (n-2)\int \sec^{n-2}(x)\tan^2(x)\,dx.$$

But now recall that $\tan^2(x) = \sec^2(x) - 1$, so

$$\begin{aligned}
\int \sec^n(x)\,dx &= \tan(x)\sec^{n-2}(x) - (n-2)\int \sec^{n-2}(x)[\sec^2(x) - 1]\,dx \\
&= \tan(x)\sec^{n-2}(x) - (n-2)\int \sec^n(x)\,dx + (n-2)\int \sec^{n-2}(x)\,dx.
\end{aligned}$$

Since the original integral has reoccurred, we can write

$$\begin{aligned}
(n-1)\int \sec^n(x)\,dx &= \tan(x)\sec^{n-2}(x) + (n-2)\int \sec^{n-2}(x)\,dx \\
\int \sec^n(x)\,dx &= \frac{\tan(x)\sec^{n-2}(x)}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx,
\end{aligned}$$

as required.

[2] (b) Using the reduction formula, we have

$$\begin{aligned}\int \sec^6(x) &= \frac{\tan(x)\sec^4(x)}{5} + \frac{4}{5}\int \sec^4(x)\,dx \\ &= \frac{1}{5}\tan(x)\sec^4(x) + \frac{4}{5}\left[\frac{\tan(x)\sec^2(x)}{3} + \frac{2}{3}\int \sec^2(x)\,dx\right] \\ &= \frac{1}{5}\tan(x)\sec^4(x) + \frac{4}{15}\tan(x)\sec^2(x) + \frac{8}{15}\int \sec^2(x)\,dx \\ &= \frac{1}{5}\tan(x)\sec^4(x) + \frac{4}{15}\tan(x)\sec^2(x) + \frac{8}{15}\tan(x) + C.\end{aligned}$$

[3] 3. (a) We use u-substitution. Let $u = x^3$ so $\frac{1}{3}\,du = x^2\,dx$. The integral becomes

$$\begin{aligned}\int x^2\cos(x^3)\,dx &= \frac{1}{3}\int \cos(u)\,du \\ &= \frac{1}{3}\sin(u) + C \\ &= \frac{1}{3}\sin(x^3) + C.\end{aligned}$$

[5] (b) Again, we use u-substitution. Let $u = x^2$ so $\frac{1}{2}\,du = x\,dx$. Then

$$\int x^3\cos(x^2)\,dx = \int x^2\cos(x^2)\cdot x\,dx = \frac{1}{2}\int u\cos(u)\,du.$$

Now we need integration by parts. Let $w = u$ so $dw = du$. Let $dv = \cos(u)\,du$ so $v = \sin(u)$. Now we can write

$$\begin{aligned}\int x^3\cos(x^2)\,dx &= \frac{1}{2}\left[u\sin(u) - \int \sin(u)\,du\right] \\ &= \frac{1}{2}u\sin(u) + \frac{1}{2}\cos(u) + C \\ &= \frac{1}{2}x^2\sin(x^2) + \frac{1}{2}\cos(x^2) + C.\end{aligned}$$

[6] (c) Let $u = \sqrt{x}$ so $du = \dfrac{1}{2\sqrt{x}}\,dx$. Since we do not have an expression of the form $\dfrac{1}{\sqrt{x}}$ in the integrand, we can write $du = \dfrac{1}{2u}\,dx$ and so $2u\,du = dx$. The integral becomes

$$\int \sqrt{x}\cos\left(\sqrt{x}\right)\,dx = \int u\cos(u)\cdot 2u\,du = 2\int u^2\cos(u)\,du.$$

Now we use integration by parts. Let $w = u^2$ so $dw = 2u\,du$. Let $dv = \cos(u)\,du$ so $v = \sin(u)$. We can write

$$\int u^2\cos(u)\,du = u^2\sin(u) - 2\int u\sin(u)\,du.$$

We need integration by parts a second time. Let $w = u$ so $dw = du$. Let $dv = \sin(u)\,du$ so $v = -\cos(u)$. Then

$$\int u\sin(u)\,du = -u\cos(u) + \int \cos(u)\,du = -u\cos(u) + \sin(u) + C$$

so

$$\int u^2\cos(u)\,du = u^2\sin(u) - 2[-u\cos(u) + \sin(u)] + C = u^2\sin(u) + 2u\cos(u) - 2\sin(u) + C,$$

and finally

$$\begin{aligned}\int \sqrt{x}\cos\left(\sqrt{x}\right)\,dx &= 2[u^2\sin(u) + 2u\cos(u) - 2\sin(u)] + C \\ &= 2x\sin\left(\sqrt{x}\right) + 4\sqrt{x}\cos\left(\sqrt{x}\right) - 4\sin\left(\sqrt{x}\right) + C.\end{aligned}$$

[2] (d) We simply write

$$\int \frac{1}{\cos^2(x)}\,dx = \int \sec^2(x)\,dx = \tan(x) + C.$$