MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Test 2	**MATHEMATICS 1001**	Fall 2019

SOLUTIONS

[7] 1. (a) We use a regular partition with subintervals of width

$$\Delta x = \frac{2-(-1)}{n} = \frac{3}{n}.$$

We choose the sample point

$$x_i = a_i = -1 + i\Delta x = -1 + \frac{3i}{n}.$$

Thus

$$\begin{aligned} f(x_i) &= 7 + 4\left(-1+\frac{3i}{n}\right) - 4\left(-1+\frac{3i}{n}\right)^2 \\ &= 7 - 4 + \frac{12i}{n} - 3 + \frac{18i}{n} - \frac{27i^2}{n^2} \\ &= \frac{30i}{n} - \frac{27i^2}{n^2}. \end{aligned}$$

Now we can write

$$\begin{aligned} \int_{-1}^{2} (7+4x-3x^2)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{30i}{n}-\frac{27i^2}{n^2}\right)\cdot\frac{3}{n} \\ &= \lim_{n\to\infty}\left[\frac{90}{n^2}\sum_{i=1}^{n} i - \frac{81}{n^3}\sum_{i=1}^{n} i^2\right] \\ &= \lim_{n\to\infty}\left[\frac{90}{n^2}\cdot\frac{n(n+1)}{2} - \frac{81}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right] \\ &= 45 - 27 \\ &= 18. \end{aligned}$$

[3] (b) We have

$$\begin{aligned} \int_{-1}^{2} (7+4x-3x^2)\,dx &= \Big[7x + 2x^2 - x^3\Big]_{-1}^{2} \\ &= (14+8-8) - (-7+2+1) \\ &= 18. \end{aligned}$$

[5] 2. (a) One approach is to factor 4 out of the square root, so

$$\begin{aligned}
\int_{\frac{1}{4}}^{\frac{\sqrt{3}}{4}} \frac{1}{\sqrt{1-4x^2}}\,dx &= \frac{1}{\sqrt{4}} \int_{\frac{1}{4}}^{\frac{\sqrt{3}}{4}} \frac{1}{\sqrt{\frac{1}{4}-x^2}}\,dx \\
&= \frac{1}{2} \int_{\frac{1}{4}}^{\frac{\sqrt{3}}{4}} \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2-x^2}}\,dx \\
&= \frac{1}{2}\left[\arcsin\left(\frac{x}{\frac{1}{2}}\right)\right]_{\frac{1}{4}}^{\frac{\sqrt{3}}{4}} \\
&= \frac{1}{2}\Big[\arcsin(2x)\Big]_{\frac{1}{4}}^{\frac{\sqrt{3}}{4}} \\
&= \frac{1}{2}\left[\arcsin\left(\frac{\sqrt{3}}{2}\right) - \arcsin\left(\frac{1}{2}\right)\right] \\
&= \frac{1}{2}\left(\frac{\pi}{3}-\frac{\pi}{6}\right) \\
&= \frac{\pi}{12}.
\end{aligned}$$

Alternatively, we could write

$$\int_{\frac{1}{4}}^{\frac{\sqrt{3}}{4}} \frac{1}{\sqrt{1-4x^2}}\,dx = \int_{\frac{1}{4}}^{\frac{\sqrt{3}}{4}} \frac{1}{\sqrt{1-(2x)^2}}\,dx.$$

Now we let $u = 2x$ so $\frac{1}{2}\,du = dx$. When $x = \frac{1}{4}$, $u = \frac{1}{2}$. When $x = \frac{\sqrt{3}}{4}$, $u = \frac{\sqrt{3}}{2}$. Thus the integral becomes

$$\begin{aligned}
\int_{\frac{1}{4}}^{\frac{\sqrt{3}}{4}} \frac{1}{\sqrt{1-4x^2}}\,dx &= \frac{1}{2} \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1-u^2}}\,du \\
&= \frac{1}{2}\Big[\arcsin(u)\Big]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \\
&= \frac{1}{2}\left[\arcsin\left(\frac{\sqrt{3}}{2}\right) - \arcsin\left(\frac{1}{2}\right)\right] \\
&= \frac{1}{2}\left(\frac{\pi}{3}-\frac{\pi}{6}\right) \\
&= \frac{\pi}{12}.
\end{aligned}$$

[6] (b) Let $u = 1-4x^2$ so $du = -8x\,dx$ and $-\frac{1}{8}\,du = x\,dx$. When $x = \frac{1}{4}$, $u = \frac{3}{4}$. When $x = \frac{\sqrt{3}}{4}$,

$u = \frac{1}{4}$. The integral becomes

$$\begin{aligned}\int_{\frac{1}{4}}^{\frac{\sqrt{3}}{4}} \frac{x}{\sqrt{1-4x^2}}\,dx &= -\frac{1}{8}\int_{\frac{3}{4}}^{\frac{1}{4}} \frac{1}{\sqrt{u}}\,du \\ &= -\frac{1}{8}\Big[2\sqrt{u}\Big]_{\frac{3}{4}}^{\frac{1}{4}} \\ &= -\frac{1}{8}\left(2\cdot\frac{1}{2} - 2\cdot\frac{\sqrt{3}}{2}\right) \\ &= \frac{\sqrt{3}-1}{8}.\end{aligned}$$

[5] (c) We can write

$$\begin{aligned}\int_{-2}^{6} f(x)\,dx &= \int_{-2}^{1} f(x)\,dx + \int_{1}^{6} f(x)\,dx \\ &= \int_{-2}^{1} (x+5)\,dx + \int_{1}^{6} 3x^{-2}\,dx \\ &= \left[\frac{1}{2}x^2 + 5x\right]_{-2}^{1} - 3\left[\frac{1}{x}\right]_{1}^{6} \\ &= \left[\left(\frac{1}{2}+5\right) - (2-10)\right] - 3\left(\frac{1}{6}-1\right) \\ &= 16.\end{aligned}$$

[5] 3. We can write

$$\begin{aligned}\int_{x}^{x^4} \tan\left(\sqrt{t}\right)\,dt &= \int_{x}^{0} \tan\left(\sqrt{t}\right)\,dt + \int_{0}^{x^4} \tan\left(\sqrt{t}\right)\,dt \\ &= -\int_{0}^{x} \tan\left(\sqrt{t}\right)\,dt + \int_{0}^{x^4} \tan\left(\sqrt{t}\right)\,dt.\end{aligned}$$

Then, by the Second Fundamental Theorem of Calculus and the Chain Rule, we have

$$\begin{aligned}g'(x) &= -\tan\left(\sqrt{x}\right) + \tan\left(\sqrt{x^4}\right)\cdot[x^4]' \\ &= 4x^3\tan(x^2) - \tan\left(\sqrt{x}\right).\end{aligned}$$

[9] 4. (a) The sketch of R is given in Figure 1. Note that $y = 12 - x$ and $y = \sqrt{x}$ intersect when

$$\begin{aligned}12 - x &= \sqrt{x} \\ (12-x)^2 &= x \\ x^2 - 25x + 144 &= 0 \\ (x-16)(x-9) &= 0\end{aligned}$$

and so $x = 16$ or $x = 9$. However, substituting these values back into the original equation, we see that only $x = 9$ is a true solution. Thus these curves intersect at the point $(9, 3)$. In addition, the line $y = 2$ intersects $y = \sqrt{x}$ at $(4, 2)$ and it intersects $y = 12 - x$ at $(10, 2)$.

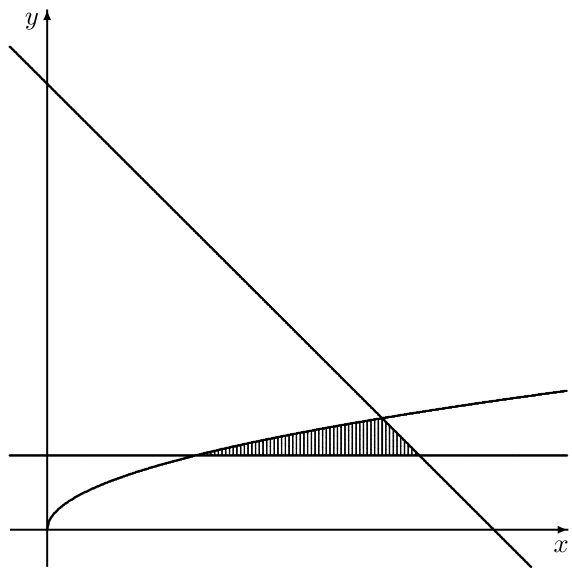

Figure 1: Question 4(a)

(b) From the graph, we can see that on the interval $[4, 9]$, the top boundary curve is $y = \sqrt{x}$ while the bottom boundary curve is the line $y = 2$. However, on the interval $[9, 10]$, the top boundary curve is $y = 12 - x$ while the bottom boundary curve remains the line $y = 2$. Hence R is not vertically simple, and its area must be represented as the sum of two integrals:

$$A = \int_4^9 \left(\sqrt{x} - 2\right)\, dx + \int_9^{10} [(12 - x) - 2]\, dx.$$

(c) The function $y = \sqrt{x}$ can be written $x = y^2$ (with $y \geq 0$), while $y = 12 - x$ becomes $x = 12 - y$. The graph shows that $x = 12 - y$ is always the rightmost boundary curve, while $x = y^2$ is always the leftmost boundary curve. So then

$$A = \int_2^3 [(12 - y) - y^2]\, dy.$$