MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Assignment 2	**MATHEMATICS 1001**	Fall 2019

SOLUTIONS

[3] 1. (a) Let $u = \ln(x)$ so $du = \dfrac{1}{x}\,dx$. Then the integral becomes

$$\begin{aligned}\int \frac{\cot(\ln(x))}{3x}\,dx &= \frac{1}{3}\int \cot(u)\,du \\ &= \frac{1}{3}\ln|\sin(u)| + C \\ &= \frac{1}{3}\ln|\sin(\ln(x))| + C.\end{aligned}$$

[6] (b) We can write

$$\int \left[x^3\sin(x^4) - \frac{\cos\left(\frac{1}{x^4}\right)}{x^5}\right] dx = \int x^3\sin(x^4)\,dx - \int \frac{\cos\left(\frac{1}{x^4}\right)}{x^5}\,dx$$

and evaluate the two integrals separately. For the first integral, let $u = x^4$ so $du = 4x^3\,dx$ and $\frac{1}{4}\,du = x^3\,dx$. Then

$$\int x^3\sin(x^4)\,dx = \frac{1}{4}\int \sin(u)\,du = \frac{1}{4}\cos(u) + C = \frac{1}{4}\cos(x^4) + C.$$

For the second integral, let $z = x^{-4}$ so $dz = -4x^{-5}\,dx$ and $-\frac{1}{4}\,dz = x^{-5}\,dx$. Then

$$\int \frac{\cos\left(\frac{1}{x^4}\right)}{x^5}\,dx = -\frac{1}{4}\int \cos(z)\,dz = -\frac{1}{4}\sin(z)+C = -\frac{1}{4}\sin(x^{-4})+C = -\frac{1}{4}\sin\left(\frac{1}{x^4}\right)+C.$$

Thus

$$\begin{aligned}\int \left[x^3\sin(x^4) - \frac{\cos\left(\frac{1}{x^4}\right)}{x^5}\right] dx &= \frac{1}{4}\cos(x^4) - \left[-\frac{1}{4}\sin\left(\frac{1}{x^4}\right)\right] + C \\ &= \frac{1}{4}\cos(x^4) + \frac{1}{4}\sin\left(\frac{1}{x^4}\right) + C.\end{aligned}$$

[5] (c) Let $u = x^2 - 3$ so $du = 2x\,dx$ and $\frac{1}{2}\,du = x\,dx$. Then we can write

$$\int x^7\sqrt{x^2-3}\,dx = \int x^6\sqrt{x^2-3}\cdot x\,dx = \frac{1}{2}\int x^6\sqrt{u}\,du.$$

Now we need to write x^6 in terms of u, so we observe that

$$x^2 = u + 3 \quad\Longrightarrow\quad x^6 = (u+3)^3.$$

Thus

$$\begin{aligned}\int x^7\sqrt{x^2-3}\,dx &= \frac{1}{2}\int (u+3)^3\sqrt{u}\,du\\ &= \frac{1}{2}\int (u^3+9u^2+27u+27)\sqrt{u}\,du\\ &= \frac{1}{2}\int (u^{\frac{7}{2}}+9u^{\frac{5}{2}}+27u^{\frac{3}{2}}+27u^{\frac{1}{2}})\,du\\ &= \frac{1}{2}\left[\frac{2}{9}u^{\frac{9}{2}}+9\cdot\frac{2}{7}u^{\frac{7}{2}}+27\cdot\frac{2}{5}u^{\frac{5}{2}}+27\cdot\frac{2}{3}u^{\frac{3}{2}}\right]+C\\ &= \frac{1}{9}(x^2-3)^{\frac{9}{2}}+\frac{9}{7}(x^2-3)^{\frac{7}{2}}+\frac{27}{5}(x^2-3)^{\frac{5}{2}}+9(x^2-3)^{\frac{3}{2}}+C.\end{aligned}$$

[6] 2. (a) We can write

$$\begin{aligned}\int \frac{x^2+4}{x\sqrt{x^2-4}}\,dx &= \int \frac{x^2}{x\sqrt{x^2-4}}\,dx+\int \frac{4}{x\sqrt{x^2-4}}\,dx\\ &= \int \frac{x}{\sqrt{x^2-4}}\,dx+4\int \frac{1}{x\sqrt{x^2-4}}\,dx.\end{aligned}$$

For the first integral, let $u=x^2-4$ so $du=2x\,dx$ and $\frac{1}{2}\,du=x\,dx$. Then

$$\int \frac{x}{\sqrt{x^2-4}}\,dx=\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du=\frac{1}{2}\int u^{-\frac{1}{2}}\,du=\frac{1}{2}\cdot 2u^{\frac{1}{2}}+C=\sqrt{x^2-4}+C.$$

The second integral is an elementary arcsecant integral:

$$\int \frac{1}{x\sqrt{x^2-4}}\,dx=\frac{1}{2}\operatorname{arcsec}\left(\frac{x}{2}\right)+C.$$

Therefore

$$\begin{aligned}\int \frac{x^2+4}{x\sqrt{x^2-4}}\,dx &= \sqrt{x^2-4}+4\left[\frac{1}{2}\operatorname{arcsec}\left(\frac{x}{2}\right)\right]+C\\ &= \sqrt{x^2-4}+2\operatorname{arcsec}\left(\frac{x}{2}\right)+C.\end{aligned}$$

[4] (b) Observe that

$$\int \frac{e^t}{\sqrt{7-e^{2t}}}\,dt=\int \frac{e^t}{\sqrt{7-(e^t)^2}}\,dt.$$

So let $u = e^t$ and $du = e^t\,dt$. The integral becomes

$$\int \frac{e^t}{\sqrt{7-e^{2t}}}\,dt = \int \frac{1}{\sqrt{7-u^2}}\,du$$
$$= \arcsin\left(\frac{u}{\sqrt{7}}\right) + C$$
$$= \arcsin\left(\frac{e^t}{\sqrt{7}}\right) + C$$
$$= \arcsin\left(\frac{\sqrt{7}e^t}{7}\right) + C.$$

[5] 3. (a) First we complete the square on the denominator:

$$16x^2 - 24x + 34 = 16\left[x^2 - \frac{3}{2}x + \frac{17}{8}\right]$$
$$= 16\left[\left(x^2 - \frac{3}{2}x + \frac{9}{16}\right) + \frac{17}{8} - \frac{9}{16}\right]$$
$$= 16\left[\left(x - \frac{3}{4}\right)^2 + \frac{25}{16}\right]$$
$$= 16\left(x - \frac{3}{4}\right)^2 + 25$$
$$= (4x-3)^2 + 25.$$

Thus

$$\int \frac{1}{16x^2 - 24x + 34}\,dx = \frac{1}{(4x-3)^2 + 25}\,dx.$$

Now let $u = 4x - 3$ so $du = 4\,dx$ and $\frac{1}{4}\,du = dx$. The integral becomes

$$\int \frac{1}{16x^2 - 24x + 34}\,dx = \frac{1}{4}\int \frac{1}{u^2 + 25}\,du$$
$$= \frac{1}{4}\cdot\frac{1}{5}\arctan\left(\frac{u}{5}\right) + C$$
$$= \frac{1}{20}\arctan\left(\frac{4x-3}{5}\right) + C.$$

[7] (b) By long division, we have:

$$\begin{array}{rl}
 & x^3 - 2x - 3 \\
x^2 + 4 \!\!\!\! & \overline{\big)\, 3x^5 + 10x^3 - 3x^2 \qquad - 12} \\
 & \underline{3x^5 + 12x^3} \\
 & \quad -2x^3 - 3x^2 \qquad - 12 \\
 & \quad \underline{-2x^3 \qquad - 8x} \\
 & \qquad\quad -3x^2 + 8x - 12 \\
 & \qquad\quad \underline{-3x^2 \qquad - 12} \\
 & \qquad\qquad\quad 8x
\end{array}$$

Thus we can write

$$\frac{3x^5 + 10x^3 - 3x^2 - 12}{x^2 + 4} = 3x^3 - 2x - 3 + \frac{8x}{x^2 + 4}.$$

Hence

$$\begin{aligned}
\int \frac{3x^5 + 10x^3 - 3x^2 - 12}{x^2 + 4}\, dx &= \int (3x^3 - 2x - 3)\, dx + 8 \int \frac{x}{x^2 + 4}\, dx \\
&= \frac{3}{4}x^4 - x^2 - 3x + 8 \int \frac{x}{x^2 + 4}\, dx.
\end{aligned}$$

For the remaining integral, we can let $u = x^2 + 4$ so $du = 2x\, dx$ and $\frac{1}{2}\, du = x\, dx$. Thus

$$\begin{aligned}
\int \frac{3x^5 + 10x^3 - 3x^2 - 12}{x^2 + 4}\, dx &= \frac{3}{4}x^4 - x^2 - 3x + 8 \cdot \frac{1}{2} \int \frac{1}{u}\, du \\
&= \frac{3}{4}x^4 - x^2 - 3x + 4 \ln|u| + C \\
&= \frac{3}{4}x^4 - x^2 - 3x + 4 \ln(x^2 + 4) + C,
\end{aligned}$$

where we can drop the absolute value in the argument of the logarithm because $x^2 + 4$ must be positive for all x.

[4] (c) Observe that

$$\int \frac{20z^3 + 5z}{6z^4 + 3z^2 + 11}\, dz = 5 \int \frac{4z^3 + z}{6z^4 + 3z^2 + 11}\, dz.$$

Let $u = 6z^4 + 3z^2 + 11$ so $du = (24z^3 + 6z)\, dz = 6(4z^3 + z)\, dz$ and $\frac{1}{6}\, dz = (4z^3 + z)\, dz$. The integral becomes

$$\begin{aligned}
\int \frac{20z^3 + 5z}{6z^4 + 3z^2 + 11}\, dz &= 5 \cdot \frac{1}{6} \int \frac{1}{u}\, du \\
&= \frac{5}{6} \ln|u| + C \\
&= \frac{5}{6} \ln(6z^4 + 3z^2 + 11) + C,
\end{aligned}$$

where we can drop the absolute value in the argument of the logarithm because $6z^4 + 3z^2 + 11$ must be positive for all z.