MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

INVERSE TRIGONOMETRY **Math 1001 Worksheet** FALL 2019

SOLUTIONS

1. (a) Note that

$$\int \frac{1}{\sqrt{9-36x^2}}\,dx = \int \frac{1}{\sqrt{3^2-(6x)^2}}\,dx$$

so let $u = 6x$ and $\frac{1}{6}\,du = dx$. The integral becomes

$$\int \frac{1}{\sqrt{9-36x^2}}\,dx = \frac{1}{6}\int \frac{1}{\sqrt{3^2-u^2}}\,du = \frac{1}{6}\left[\arcsin\left(\frac{u}{3}\right)\right] + C$$
$$= \frac{1}{6}\arcsin(2x) + C.$$

Alternatively, we could write

$$\int \frac{1}{\sqrt{9-36x^2}}\,dx = \int \frac{1}{\sqrt{36\left(\frac{1}{4}-x^2\right)}}\,dx = \frac{1}{6}\int \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2-x^2}}\,dx$$
$$= \frac{1}{6}\arcsin\left(\frac{x}{\frac{1}{2}}\right) + C = \frac{1}{6}\arcsin(2x) + C.$$

(b) We have

$$\int \frac{1}{-9x^2-2}\,dx = -\int \frac{1}{(3x)^2+2}\,dx$$

so let $u = 3x$ and $\frac{1}{3}\,du = dx$. Then

$$\frac{1}{-9x^2-2}\,dx = -\frac{1}{3}\int \frac{1}{u^2+(\sqrt{2})^2}\,du = -\frac{1}{3}\left[\frac{1}{\sqrt{2}}\arctan\left(\frac{u}{\sqrt{2}}\right)\right] + C$$
$$= -\frac{\sqrt{2}}{6}\arctan\left(\frac{3\sqrt{2}x}{2}\right) + C.$$

Alternatively, we could write

$$\int \frac{1}{-9x^2-2}\,dx = -\frac{1}{9}\int \frac{1}{x^2+\frac{2}{9}}\,dx = -\frac{1}{9}\cdot\frac{1}{\sqrt{\frac{2}{9}}}\arctan\left(\frac{1}{\sqrt{\frac{2}{9}}}\right) + C$$
$$= -\frac{\sqrt{2}}{6}\arctan\left(\frac{3\sqrt{2}x}{2}\right) + C.$$

(c) We first complete the square, giving

$$x^2 + 4x = (x^2 + 4x + 4) - 4 = (x + 2)^2 - 2^2$$

so we can write

$$\int \frac{1}{(x+2)\sqrt{x^2+4x}}\,dx = \int \frac{1}{(x+2)\sqrt{(x+2)^2 - 2^2}}\,dx.$$

Let $u = x + 2$ so $du = dx$. Now we have

$$\int \frac{1}{(x+2)\sqrt{x^2+4x}}\,dx = \int \frac{1}{u\sqrt{u^2-4}}\,du = \frac{1}{2}\operatorname{arcsec}\left(\frac{u}{2}\right) + C$$
$$= \frac{1}{2}\operatorname{arcsec}\left(\frac{x+2}{2}\right) + C.$$

(d) Let $u = \cos(\theta)$ so $-du = \sin(\theta)\,d\theta$. The integral becomes

$$\int \frac{\sin(\theta)}{\cos^2(\theta)+4}\,d\theta = -\int \frac{1}{u^2+2^2}\,du = -\frac{1}{2}\arctan\left(\frac{u}{2}\right) + C$$
$$= -\frac{1}{2}\arctan\left(\frac{\cos(\theta)}{2}\right) + C.$$

(e) Completing the square gives

$$\frac{1}{9}x^2 - 4x + 40 = \frac{1}{9}[x^2 - 36x + 360] = \frac{1}{9}[(x^2 - 36x + 324) + 36]$$
$$= \frac{1}{9}[(x-18)^2 + 36] = \left(\frac{x}{3} - 6\right)^2 + 2^2.$$

The integral can now be written

$$\int \frac{1}{\frac{1}{9}x^2 - 4x + 40}\,dx = \int \frac{1}{\left(\frac{x}{3} - 6\right)^2 + 2^2}\,dx.$$

Let $u = \frac{x}{3} - 6$ so $3\,du = dx$. Now we have

$$\int \frac{1}{\frac{1}{9}x^2 - 4x + 40}\,dx = 3\int \frac{1}{u^2+2^2}\,du = 3\left[\frac{1}{2}\arctan\left(\frac{u}{2}\right) + C\right]$$
$$= \frac{3}{2}\arctan\left(\frac{\frac{x}{3}-6}{2}\right) + C = \frac{3}{2}\arctan\left(\frac{x-18}{6}\right) + C.$$

(f) Let $u = e^{2x}$ so $\frac{1}{2}\,du = e^{2x}\,dx$. The integral can be written

$$\int \frac{e^{2x}}{\sqrt{-e^{4x} - 10e^{2x} - 24}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{-u^2 - 10u - 24}}.$$

But note that

$$\begin{aligned}-u^2-10u-24 &= -[u^2+10u+24] = -[(u^2+10u+25)-1] \\ &= -[(u+5)^2-1] = 1-(u+5)^2\end{aligned}$$

so the integral becomes

$$\int \frac{e^{2x}}{\sqrt{-e^{4x}-10e^{2x}-24}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{1-(u+5)^2}}.$$

Next let $v = u+5$, so $dv = du$. Now we obtain

$$\begin{aligned}\int \frac{e^{2x}}{\sqrt{-e^{4x}-10e^{2x}-24}}\,dx &= \frac{1}{2}\int \frac{1}{\sqrt{1-v^2}}\,dv = \frac{1}{2}[\arcsin(v)]+C \\ &= \frac{1}{2}\arcsin(u+5)+C = \frac{1}{2}\arcsin(e^{2x}+5)+C.\end{aligned}$$

(g) First we rewrite the integral as

$$\int \frac{9x-6}{\sqrt{1-25x^2}}\,dx = 9\int \frac{x}{\sqrt{1-25x^2}}\,dx - 6\int \frac{dx}{\sqrt{1^2-(5x)^2}}.$$

For the first integral, let $u = 1-25x^2$ so $-\frac{1}{50}\,du = x\,dx$. For the second integral, let $v = 5x$ so $\frac{1}{5}\,dv = dx$. Then the integral becomes

$$\begin{aligned}\int \frac{9x-6}{\sqrt{1-25x^2}}\,dx &= -\frac{9}{50}\int u^{-\frac{1}{2}}\,du - \frac{6}{5}\int \frac{dv}{\sqrt{1^2-v^2}} \\ &= -\frac{9}{50}[2\sqrt{u}] - \frac{6}{5}\arcsin(v)+C \\ &= -\frac{9}{25}\sqrt{1-25x^2} - \frac{6}{5}\arcsin(5x)+C.\end{aligned}$$

(h) Let $u = x^2$ so

$$\begin{aligned}4x^4+4x^2+17 = 4u^2+4u+17 &= 4\left[u^2+u+\frac{17}{4}\right] = 4\left[\left(u^2+u+\frac{1}{4}\right)+4\right] \\ &= 4\left[\left(u+\frac{1}{2}\right)^2+4\right] = (2u+1)^2+4^2 = (2x^2+1)^2+4^2.\end{aligned}$$

Now the integral can be written

$$\int \frac{2x}{4x^4+4x^2+17}\,dx = \int \frac{2x}{(2x^2+1)^2+4^2}\,dx.$$

Then set $v = 2x^2+1$ so $\frac{1}{2}\,dv = 2x\,dx$. We obtain

$$\begin{aligned}\int \frac{2x}{4x^4+4x^2+17}\,dx &= \frac{1}{2}\int \frac{dv}{v^2+4^2} = \frac{1}{2}\left[\frac{1}{4}\arctan\left(\frac{v}{4}\right)\right]+C \\ &= \frac{1}{8}\arctan\left(\frac{2x^2+1}{4}\right)+C.\end{aligned}$$