MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Test 1	**MATHEMATICS 1001**	Fall 2019

SOLUTIONS

[6] 1. (a) We let $u = 16x^2 - 25$ so $du = 32x\,dx$ and $\frac{1}{32}\,du = x\,dx$. The integral becomes

$$\begin{aligned}\int \frac{x}{\sqrt{16x^2-25}}\,dx &= \frac{1}{32}\int \frac{1}{\sqrt{u}}\,du \\ &= \frac{1}{32}\int u^{-\frac{1}{2}}\,du \\ &= \frac{1}{32}\cdot 2\sqrt{u} + C \\ &= \frac{1}{16}\sqrt{16x^2-25} + C.\end{aligned}$$

[4] (b) One approach is the factor the coefficient of 16 out of the square root, giving

$$\begin{aligned}\int \frac{1}{x\sqrt{16x^2-25}}\,dx &= \int \frac{1}{x\sqrt{16}\cdot\sqrt{x^2-\frac{25}{16}}}\,dx \\ &= \frac{1}{4}\int \frac{1}{x\sqrt{x^2-\left(\frac{5}{4}\right)^2}}\,dx \\ &= \frac{1}{4}\cdot\frac{1}{\left(\frac{5}{4}\right)}\operatorname{arcsec}\left(\frac{x}{\left(\frac{5}{4}\right)}\right) + C \\ &= \frac{1}{5}\operatorname{arcsec}\left(\frac{4}{5}x\right) + C.\end{aligned}$$

Alternatively, we could observe that

$$\int \frac{1}{x\sqrt{16x^2-25}}\,dx = \int \frac{1}{x\sqrt{(4x)^2-5^2}}\,dx.$$

We let $u = 4x$ so $du = 4\,dx$ and $\frac{1}{4}\,du = dx$. Furthermore, $x = \frac{1}{4}u$. Thus the integral becomes

$$\begin{aligned}\int \frac{1}{x\sqrt{16x^2-25}}\,dx &= \int \frac{1}{\left(\frac{1}{4}u\right)\sqrt{u^2-5^2}}\cdot\frac{1}{4}\,du \\ &= \int \frac{1}{u\sqrt{u^2-5^2}}\,du \\ &= \frac{1}{5}\operatorname{arcsec}\left(\frac{u}{5}\right) + C \\ &= \frac{1}{5}\operatorname{arcsec}\left(\frac{4x}{5}\right) + C.\end{aligned}$$

[5] (c) We use integration by parts with $w = x$ so $dw = dx$, and $dv = e^{4x-5}\,dx$ so $v = \frac{1}{4}e^{4x-5}$. Thus we have

$$\begin{aligned}\int xe^{4x-5}\,dx &= \frac{1}{4}xe^{4x-5} - \frac{1}{4}\int e^{4x-5}\,dx \\ &= \frac{1}{4}xe^{4x-5} - \frac{1}{4}\cdot\frac{1}{4}e^{4x-5} + C \\ &= \frac{1}{4}xe^{4x-5} - \frac{1}{16}e^{4x-5} + C.\end{aligned}$$

[25] 2. (a) Since $\sin^2(5x) + \cos^2(5x) = 1$, we can write

$$\begin{aligned}\int \frac{\sin(5x)}{\sqrt{1-\sin^2(5x)}}\,dx &= \int \frac{\sin(5x)}{\sqrt{\cos^2(5x)}}\,dx \\ &= \int \frac{\sin(5x)}{\cos(5x)}\,dx \\ &= \int \tan(5x)\,dx \\ &= \frac{1}{5}\ln|\sec(5x)| + C = -\frac{1}{5}\ln|\cos(5x)| + C.\end{aligned}$$

(b) We complete the square:

$$\begin{aligned}9x^2 + 12 + 8 &= 9\left[x^2 + \frac{4}{3}x + \frac{8}{9}\right] \\ &= 9\left[\left(x^2 + \frac{4}{3}x + \frac{4}{9}\right) + \frac{8}{9} - \frac{4}{9}\right] \\ &= 9\left[\left(x + \frac{2}{3}\right)^2 + \frac{4}{9}\right] \\ &= (3x+2)^2 + 4.\end{aligned}$$

Now the integral can be written

$$\int \frac{1}{9x^2 + 12x + 8}\,dx = \int \frac{1}{(3x+2)^2 + 4}\,dx.$$

Let $u = 3x + 2$ so $du = 3\,dx$ and $\frac{1}{3}\,du = dx$. The integral becomes

$$\begin{aligned}\int \frac{1}{9x^2 + 12x + 8}\,dx &= \frac{1}{3}\int \frac{1}{u^2 + 4}\,du \\ &= \frac{1}{3}\cdot\frac{1}{2}\arctan\left(\frac{u}{2}\right) + C \\ &= \frac{1}{6}\arctan\left(\frac{3x+2}{2}\right) + C.\end{aligned}$$

(c) We let $u = 2x^2$ so $du = 4x\,dx$ and $\frac{1}{4}\,du = x\,dx$. Furthermore, $x^2 = \frac{1}{2}u$. Now we can write

$$\begin{aligned}\int x^3 \sin(2x^2)\,dx &= \int x^2 \sin(2x^2) \cdot x\,dx \\ &= \int \left(\frac{1}{2}u\right) \sin(u) \cdot \frac{1}{4}\,du \\ &= \frac{1}{8} \int u \sin(u)\,du.\end{aligned}$$

Next we use integration by parts with $w = u$ so $dw = du$, and $dv = \sin(u)\,du$ so $v = -\cos(u)$. Thus we obtain

$$\begin{aligned}\int x^3 \sin(2x^2)\,dx &= \frac{1}{8}\left[-u\cos(u) + \int \cos(u)\,du\right] \\ &= -\frac{1}{8}u\cos(u) + \frac{1}{8}\sin(u) + C \\ &= -\frac{1}{8}(2x^2)\cos(2x^2) + \frac{1}{8}\sin(2x^2) + C \\ &= -\frac{1}{4}x^2\cos(2x^2) + \frac{1}{8}\sin(2x^2) + C.\end{aligned}$$

(d) We use integration by parts with $w = \ln^2(x)$ so $dw = \dfrac{2\ln(x)}{x}\,dx$, and $dv = x^2\,dx$ so $v = \frac{1}{3}x^3$. This yields

$$\begin{aligned}\int x^2 \ln^2(x)\,dx &= \frac{1}{3}x^3\ln^2(x) - \int \left(\frac{1}{3}x^3\right) \cdot \frac{2\ln(x)}{x}\,dx \\ &= \frac{1}{3}x^3\ln^2(x) - \frac{2}{3}\int x^2 \ln(x)\,dx.\end{aligned}$$

Now we use integration by parts for a second time, with $w = \ln(x)$ so $dw = \dfrac{1}{x}\,dx$, and again $dv = x^2\,dx$ so $v = \frac{1}{3}x^3$. Hence we have

$$\begin{aligned}\int x^2 \ln^2(x)\,dx &= \frac{1}{3}x^3\ln^2(x) - \frac{2}{3}\left[\frac{1}{3}x^3\ln(x) - \int \left(\frac{1}{3}x^3\right) \cdot \frac{1}{x}\,dx\right] \\ &= \frac{1}{3}x^3\ln^2(x) - \frac{2}{9}x^3\ln(x) + \frac{2}{9}\int x^2\,dx \\ &= \frac{1}{3}x^3\ln^2(x) - \frac{2}{9}x^3\ln(x) + \frac{2}{9}\cdot\frac{1}{3}x^3 + C \\ &= \frac{1}{3}x^3\ln^2(x) - \frac{2}{9}x^3\ln(x) + \frac{2}{27}x^3 + C.\end{aligned}$$

(e) We let $u = \ln(x)$ so $du = \dfrac{1}{x}\,dx$. The integral becomes

$$\begin{aligned}
\int \frac{1}{x\sqrt{9 - \ln^2(x)}}\,dx &= \int \frac{1}{\sqrt{3^2 - \ln^2(x)}} \cdot \frac{1}{x}\,dx \\
&= \int \frac{1}{\sqrt{3^2 - u^2}}\,du \\
&= \arcsin\left(\frac{u}{3}\right) + C \\
&= \arcsin\left(\frac{\ln(x)}{3}\right) + C.
\end{aligned}$$

(f) Here we can use long division:

$$\begin{array}{r@{\,}r}
 & 2x - 4 \\ \cline{2-2}
2x^2 + 1 \big) & 4x^3 - 8x^2 + 5x - 4 \\
 & \underline{4x^3 \qquad\quad + 2x \quad\;\;} \\
 & -8x^2 + 3x - 4 \\
 & \underline{-8x^2 \qquad\quad - 4} \\
 & 3x \quad\;\;
\end{array}$$

Thus

$$\begin{aligned}
\int \frac{4x^3 - 8x^2 + 5x - 4}{2x^2 + 1}\,dx &= \int \left(2x - 4 + \frac{3x}{2x^2 + 1}\right)\,dx \\
&= x^2 - 4x + 3\int \frac{x}{2x^2 + 1}\,dx.
\end{aligned}$$

We let $u = 2x^2 + 1$ so $du = 4x\,dx$ and $\frac{1}{4}\,du = x\,dx$. Thus

$$\begin{aligned}
\int \frac{4x^3 - 8x^2 + 5x - 4}{2x^2 + 1}\,dx &= x^2 - 4x + \frac{3}{4}\int \frac{1}{u}\,du \\
&= x^2 - 4x + \frac{3}{4}\ln|u| + C \\
&= x^2 - 4x + \frac{3}{4}\ln(2x^2 + 1) + C.
\end{aligned}$$