MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

VOLUMES | **Math 1001 Worksheet** | FALL 2019

SOLUTIONS

1. (a) i. The axis of symmetry is a horizontal line, so we integrate with respect to x. The region being rotated is above the axis, with the boundary curve $y = \sqrt{x-2}$ farther from the axis than the boundary curve $y = 0$. Hence the outer radius is $R(x) = \sqrt{x-2}$ and the inner radius is $r(x) = 0$. Thus

$$\begin{aligned} V &= \pi \int_2^6 ([\sqrt{x-2}]^2 - [0]^2)\, dx \\ &= \pi \int_2^6 (x-2)\, dx \\ &= \pi \left[\frac{1}{2}x^2 - 2x\right]_2^6 \\ &= \pi[(18-12)-(2-4)] \\ &= 8\pi. \end{aligned}$$

ii. The axis of symmetry is a vertical line, so we integrate with respect to y. The region being rotated is to the right of the axis, with the boundary curve $x = 6$ farther from the axis than the boundary curve $y = \sqrt{x-2}$, which we can rewrite as $x = y^2 + 2$. Hence the outer radius is $R(y) = 6$ and the inner radius is $r(y) = y^2 + 2$. Finally, note that when $x = 6$, $y = \sqrt{x-2} = 2$ so the interval over which we integrate is $[0, 2]$. Thus

$$\begin{aligned} V &= \pi \int_0^2 ([6]^2 - [y^2+2]^2)\, dy \\ &= \pi \int_0^2 (32 - 4y^2 - y^4)\, dy \\ &= \pi \left[32y - \frac{4}{3}y^3 - \frac{1}{5}y^5\right]_0^2 \\ &= \pi \left[64 - \frac{32}{3} - \frac{32}{5}\right] \\ &= \frac{704\pi}{15}. \end{aligned}$$

iii. The axis of symmetry is a vertical line, so we integrate with respect to y. The region being rotated is to the left of the axis, with the boundary curve $x = y^2 + 2$

farther from the axis than the boundary curve $y = 6$. Hence the outer radius is $R(y) = 6 - (y^2 - 2) = 4 - y^2$ and the inner radius is $r(y) = 6 - 6 = 0$. Thus

$$\begin{aligned} V &= \pi \int_0^2 ([4 - y^2]^2 - [0]^2)\, dy \\ &= \pi \int_0^2 (16 - 8y^2 + y^4)\, dy \\ &= \pi \left[16y - \frac{8}{3}y^3 + \frac{1}{5}y^5\right]_0^2 \\ &= \pi \left[32 - \frac{64}{3} + \frac{32}{5}\right] \\ &= \frac{256\pi}{15}. \end{aligned}$$

(b) i. The axis of symmetry is a horizontal line, so we integrate with respect to x. The region being rotated is below the axis, with the boundary curve $y = x^2$ farther from the axis than the boundary curve $y = 2x$. Hence the outer radius is $R(x) = 5 - x^2$ and the inner radius is $r(x) = 5 - 2x$. Also, we find the endpoints of the interval of integration by setting

$$x^2 = 2x \quad \Longrightarrow \quad x^2 - 2x = x(x - 2) = 0$$

giving $x = 0$ and $x = 2$. Thus

$$\begin{aligned} V &= \pi \int_0^2 ([5 - x^2]^2 - [5 - 2x]^2)\, dx \\ &= \pi \int_0^2 (x^4 - 14x^2 + 20x)\, dx \\ &= \pi \left[\frac{1}{5}x^5 - \frac{14}{3}x^3 + 10x^2\right]_0^2 \\ &= \pi \left[\frac{32}{5} - \frac{112}{3} + 40\right] \\ &= \frac{136\pi}{15}. \end{aligned}$$

ii. The axis of symmetry is a horizontal line, so we integrate with respect to x. The region being rotated is above the axis, with the boundary curve $y = 2x$ farther from the axis than the boundary curve $y = x^2$. Hence the outer radius is $R(x) =$

$2x - (-2) = 2x + 2$ and the inner radius is $r(x) = x^2 - (-2) = x^2 + 2$. Thus

$$\begin{aligned} V &= \pi \int_0^2 \left([2x+2]^2 - [x^2+2]^2\right) dx \\ &= \pi \int_0^2 (8x - x^4)\, dx \\ &= \pi \left[4x^2 - \frac{1}{5}x^5\right]_0^2 \\ &= \pi \left[16 - \frac{32}{5}\right] \\ &= \frac{48\pi}{5}. \end{aligned}$$

iii. The axis of symmetry is a vertical line, so we integrate with respect to y. The region being rotated is to the right of the axis, with the boundary curve $y = x^2$ (which we can rewrite as $x = \sqrt{y}$) farther from the axis than the boundary curve $y = 2x$ (which we can rewrite as $x = \frac{1}{2}y$). Hence the outer radius is $R(y) = \sqrt{y}$ and the inner radius is $r(y) = \frac{1}{2}y$. Finally, note that $x = 0$ corresponds to $y = 0$ while $x = 2$ corresponds to $y = 4$, so the interval over which we integrate is $[0, 4]$. Thus

$$\begin{aligned} V &= \pi \int_0^4 \left([\sqrt{y}]^2 - \left[\frac{1}{2}y\right]^2\right) dy \\ &= \pi \int_0^4 \left(y - \frac{1}{4}y^2\right) dy \\ &= \pi \left[\frac{1}{2}y^2 - \frac{1}{12}y^3\right]_0^4 \\ &= \pi \left[8 - \frac{16}{3}\right] \\ &= \frac{8\pi}{3}. \end{aligned}$$

(c) i. The axis of symmetry is a vertical line, so we integrate with respect to y. The region being rotated is to the right of the axis, with the boundary curve $xy = 12$ (which we can rewrite as $x = \frac{12}{y}$) farther from the axis than the boundary curve $3x - y = 0$ (which we can rewrite as $x = \frac{1}{3}y$). Hence the outer radius is $R(y) = \frac{12}{y} - (-1) = \frac{12}{y} + 1$ and the inner radius is $r(y) = \frac{1}{3}y - (-1) = \frac{1}{3}y + 1$. To find the upper boundary of the interval of integration, we set

$$\frac{12}{y} = \frac{1}{3}y \implies y^2 = 36$$

and so $y = 6$ (since the region of interest is in the first quadrant, we can ignore the solution $y - 6$). Hence we must integrate over the interval $[2, 6]$, giving

$$\begin{aligned}
V &= \pi \int_2^6 \left(\left[\frac{12}{y} + 1 \right]^2 - \left[\frac{1}{3} y + 1 \right]^2 \right) dy \\
&= \pi \int_2^6 \left(\frac{144}{y^2} + \frac{24}{y} - \frac{1}{9} y^2 - \frac{2}{3} y \right) dy \\
&= \pi \left[-\frac{144}{y} + 24 \ln|y| - \frac{1}{27} y^3 - \frac{1}{3} y^2 \right]_2^6 \\
&= \pi \left[(-24 + 24\ln(6) - 8 - 12) - \left(-72 + 24\ln(2) - \frac{8}{27} - \frac{4}{3} \right) \right] \\
&= \pi \left[\frac{800}{27} + 24\ln(3) \right].
\end{aligned}$$

ii. The axis of symmetry is a vertical line, so we integrate with respect to y. The region being rotated is to the left of the axis, with the boundary curve $x = \frac{1}{3}y$ farther from the axis than the boundary curve $x = \frac{12}{y}$. Hence the outer radius is $R(y) = 7 - \frac{1}{3}y$ and the inner radius is $r(y) = 7 - \frac{12}{y}$. Thus

$$\begin{aligned}
V &= \pi \int_2^6 \left(\left[7 - \frac{1}{3} y \right]^2 - \left[7 - \frac{12}{y} \right]^2 \right) dy \\
&= \pi \int_2^6 \left(\frac{1}{9} y^2 - \frac{14}{3} y - \frac{144}{y^2} + \frac{168}{y} \right) dy \\
&= \pi \left[\frac{1}{27} y^3 - \frac{7}{3} y^2 + \frac{144}{y} + 168 \ln|y| \right]_2^6 \\
&= \pi \left[(8 - 84 + 24 + 168\ln(6)) - \left(\frac{8}{27} - \frac{28}{3} + 72 + 168\ln(2) \right) \right] \\
&= \pi \left[168\ln(3) - \frac{3104}{27} \right].
\end{aligned}$$

iii. The axis of symmetry is a horizontal line, so we integrate with respect to x. The region being rotated is above the axis, and while the boundary curve closest to the axis is always $y = 2$ — so that $r(x) = 0$ — the farther boundary curve is $xy = 12$ (which can be written $y = \frac{12}{x}$) for some values of x, and $3x - y = 0$ (which can be rewritten $y = 3x$) for others. To determine the point where this changes, we recall from earlier that the y-value of their point of intersection is $y = 6$ which implies $x = 2$. Also, we need the left and right endpoints of the entire interval of integration. The left endpoint is the intersection of $y = 2$ and $y = 3x$, namely $x = \frac{2}{3}$. The right endpoint is the intersection of $y = 2$ and $y = \frac{12}{x}$, which is $x = 6$. Hence on $[\frac{2}{3}, 2]$ the outer radius is $R(x) = 3x - 2$ while on $[2, 6]$ the outer radius is

$R(x) = \frac{12}{x} - 2$. Hence we calculate

$$\begin{aligned}
V &= \pi \int_{\frac{2}{3}}^{2} \left([3x-2]^2 - [0]^2\right) dx + \pi \int_{2}^{6} \left(\left[\frac{12}{x} - 2\right]^2 - [0]^2 \right) dx \\
&= \pi \int_{\frac{2}{3}}^{2} (9x^2 - 12x + 4)\, dx + \pi \int_{2}^{6} \left(\frac{144}{x^2} - \frac{48}{x} + 4\right) dx \\
&= \pi \left[\left[3x^3 - 6x^2 + 4x\right]\right]_{\frac{2}{3}}^{2} + \pi \left[-\frac{144}{x} - 48\ln|x| + 4x\right]_2^6 \\
&= \pi \left[(24 - 24 + 8) - \left(\frac{8}{9} - \frac{8}{3} + \frac{8}{3}\right)\right] \\
&\quad + \pi \left[(-24 - 48\ln(6) + 24) - (-72 - 48\ln(2) + 8)\right] \\
&= \pi \left[\frac{64}{9} + 64 - 48\ln(3)\right] \\
&= \pi \left[\frac{640}{9} - 48\ln(3)\right]
\end{aligned}$$

2. (a) i. The axis of symmetry is horizontal so we integrate with respect to y. The region is above the axis, so $p(y) = y - 2$. The boundary curve $xy = 12$ (which we can write as $x = \frac{12}{y}$) is to the right of $2x - y = 0$ (which we can write as $x = \frac{1}{3}y$), so $h(y) = \frac{12}{y} - \frac{1}{3}y$. We must also determine the upper limit of the interval of integration; we set

$$\frac{12}{y} = \frac{1}{3}y \quad \Longrightarrow \quad y^2 = 36$$

giving $y = \pm 6$; in this case, only the positive root is of interest. Hence

$$\begin{aligned}
V &= 2\pi \int_2^6 (y-2)\left(\frac{12}{y} - \frac{1}{3}y\right) dy \\
&= 2\pi \int_2^6 \left[12 - \frac{1}{3}y^2 - \frac{24}{y} + \frac{2}{3}y\right] dy \\
&= 2\pi \left[12y - \frac{1}{9}y^3 - 24\ln|y| + \frac{1}{3}y^2\right]_2^6 \\
&= 2\pi \left[\frac{320}{9} - 24\ln(3)\right] \\
&= \frac{640\pi}{9} - 48\pi\ln(3).
\end{aligned}$$

ii. Again the axis (the line $x = 0$) is horizontal so we integrate with respect to y. The region is again above the axis so $p(y) = y - 0 = y$. The height function is again

$h(y) = \frac{12}{y} - \frac{1}{3}y$ so

$$\begin{aligned} V &= 2\pi \int_2^6 y\left(\frac{12}{y} - \frac{1}{3}y\right) dy \\ &= 2\pi \int_2^6 \left[12 - \frac{1}{3}y^2\right] dy \\ &= 2\pi \left[12y - \frac{1}{9}y^3\right]_2^6 \\ &= 2\pi \left[\frac{224}{9}\right] \\ &= \frac{448\pi}{9}. \end{aligned}$$

(b) i. The axis of symmetry is vertical, so we integrate with respect to x. The region is to the left of the axis so $p(x) = 6 - x$. The boundary curve $y = 4 - \frac{1}{2}x$ is above the curve $y = \sqrt{x}$ so $h(x) = 4 - \frac{1}{2}x - \sqrt{x}$. To find the upper limit of integration, we solve

$$4 - \frac{1}{2}x = \sqrt{x} \quad \Longrightarrow \quad 16 - 4x + \frac{1}{4}x^2 = x \quad \Longrightarrow \quad x^2 - 20x + 64 = (x-16)(x-4) = 0$$

so either $x = 16$ or $x = 4$. However $x = 16$ does not satisfy the original equation (it is an artefact of squaring both sides) so the desired x-value is $x = 4$. Hence

$$\begin{aligned} V &= 2\pi \int_0^4 (6-x)\left(4 - \frac{1}{2}x - \sqrt{x}\right) dx \\ &= 2\pi \int_0^4 \left[24 - 7x - 6\sqrt{x} + \frac{1}{2}x^2 + x^{\frac{3}{2}}\right] dx \\ &= 2\pi \left[24x - \frac{7}{2}x^2 - 4x^{\frac{3}{2}} + \frac{1}{6}x^3 + \frac{2}{5}x^{\frac{5}{2}}\right]_0^4 \\ &= 2\pi \left[\frac{472}{15}\right] \\ &= \frac{944\pi}{15}. \end{aligned}$$

ii. The axis of symmetry is again vertical so we integrate with respect to x. The region is above the axis this time so $p(x) = x - (-6) = x + 6$. The height function is again

$h(x) = 4 - \frac{1}{2}x - \sqrt{x}$, so

$$\begin{aligned} V &= 2\pi \int_0^4 (x+6)\left(4 - \frac{1}{2}x - \sqrt{x}\right)\,dx \\ &= 2\pi \int_0^4 \left[24 + x - 6\sqrt{x} - \frac{1}{2}x^2 - x^{\frac{3}{2}}\right]\,dx \\ &= 2\pi \left[24x + \frac{1}{2}x^2 - 4x^{\frac{3}{2}} - \frac{1}{6}x^3 - \frac{2}{5}x^{\frac{5}{2}}\right]_0^4 \\ &= 2\pi \left[\frac{728}{15}\right] \\ &= \frac{1456\pi}{15}. \end{aligned}$$

(c) i. The axis of symmetry is vertical (the line $x = 0$) so we integrate with respect to x. The region lies to the left of the axis so $p(x) = 0 - x = -x$. The boundary curve $y = -x^2$ lies above the curve $x = -\frac{1}{8}y^2$. We must rewrite the latter as a function of x. Solving for y gives $y = \pm\sqrt{-8x}$ but only the negative root forms part of the boundary of the region; hence $h(x) = -x^2 - (-\sqrt{-8x}) = \sqrt{-8x} - x^2$. We must also determine the limits of integration, so we set

$$-x^2 = -\sqrt{-8x} \quad\Longrightarrow\quad x^4 = -8x \quad\Longrightarrow\quad x(x^3 + 8) = 0$$

so $x = 0$ and $x = -2$. Thus

$$\begin{aligned} V &= 2\pi \int_{-2}^0 (-x)(\sqrt{-8x} - x^2)\,dx \\ &= 2\pi \int_{-2}^0 \left[x^3 - x\sqrt{-8x}\right]\,dx \\ &= 2\pi \int_{-2}^0 x^3\,dx - 2\pi \int_{-2}^0 x\sqrt{-8x}\,dx. \end{aligned}$$

To evaluate the second integral, one way to proceed is to let $u = -8x$ so $x = -\frac{1}{8}u$ and $-\frac{1}{8}\,du = dx$. Also, $x = -2$ becomes $u = 16$ while $x = 0$ becomes $u = 0$. Hence we have

$$
\begin{aligned}
V &= 2\pi \int_{-2}^{0} x^3\,dx - \frac{\pi}{32}\int_{16}^{0} u\sqrt{u}\,du \\
&= 2\pi \int_{-2}^{0} x^3\,dx - \frac{\pi}{32}\int_{16}^{0} u^{\frac{3}{2}}\,du \\
&= 2\pi \left[\frac{1}{4}x^4\right]_{-2}^{0} - \frac{\pi}{32}\left[\frac{2}{5}u^{\frac{5}{2}}\right]_{16}^{0} \\
&= 2\pi[-4] - \frac{\pi}{32}\left[-\frac{2048}{5}\right] \\
&= \frac{24\pi}{5}.
\end{aligned}
$$

ii. The axis of symmetry is horizontal (the line $y = 0$) so we integrate with respect to y. The region is below the axis so $p(y) = 0 - y = -y$. The boundary curve $x = -\frac{1}{8}y^2$ lies to the right of the curve $y = -x^2$ which, following the same rationale as above, we can rewrite as $x = -\sqrt{-y}$ so $h(y) = -\frac{1}{8}y^2 - (-\sqrt{-y}) = \sqrt{-y} - \frac{1}{8}y^2$. To find the limits of integration, the easiest method is to substitute the y-values found in part (a) back into one of the boundary curve equation. Hence we see that $x = -2$ corresponds to $y = -4$ while $x = 0$ gives $y = 0$. We calculate

$$
\begin{aligned}
V &= 2\pi \int_{-4}^{0} (-y)\left(\sqrt{-y} - \frac{1}{8}y^2\right)\,dy \\
&= 2\pi \int_{-4}^{0} \left[\frac{1}{8}y^3 - y\sqrt{-y}\right]\,dy \\
&= 2\pi \int_{-4}^{0} \left[\frac{1}{8}y^3 + (-y)^{\frac{3}{2}}\right]\,dy \\
&= 2\pi \left[\frac{1}{32}y^4 - \frac{2}{5}(-y)^{\frac{5}{2}}\right]_{-4}^{0} \\
&= 2\pi\left[\frac{24}{5}\right] \\
&= \frac{48\pi}{5}.
\end{aligned}
$$

3. (a) The plane shape which will generate the desired cone is a right triangle with base a and height b. We can form such a plane shape by considering the region bounded by the curves $y = -\frac{b}{a}x + b$ (which can also be written as $x = -\frac{a}{b}y + a$), $y = 0$ and $x = 0$. The cone is then obtained by rotating this region about the y-axis.

To use the disc-method, we must integrate with respect to y. The outer radius is $R(y) =$

$\left(-\frac{a}{b}y + a\right) - 0 = -\frac{a}{b}y + a$ and the inner radius is $r(y) = 0 - 0$. Thus we calculate

$$\begin{aligned} V &= \pi \int_0^b \left[-\frac{a}{b}y + a\right]^2 dy \\ &= \pi \int_0^b \left[\frac{a^2}{b^2}y^2 - \frac{2a^2}{b}y + a^2\right] dy \\ &= \pi \left[\frac{a^2}{3b^2}y^3 - \frac{a^2}{b}y^2 + a^2 y\right]_0^b \\ &= \frac{\pi}{3}a^2 b. \end{aligned}$$

(b) To use the shell method we must integrate with respect to x. The average radius is $p(x) = x - 0 = x$ and the height is $h(x) = \left(-\frac{b}{a}x + b\right) - 0 = -\frac{b}{a}x + b$. So we compute

$$\begin{aligned} V &= 2\pi \int_0^a x\left(-\frac{b}{a}x + b\right) dx \\ &= 2\pi \int_0^a \left(-\frac{b}{a}x^2 + bx\right) dx \\ &= 2\pi \left[-\frac{b}{3a}x^3 + \frac{b}{2}x^2\right]_0^a \\ &= 2\pi \left[\frac{1}{6}a^2 b\right] \\ &= \frac{\pi}{3}a^2 b. \end{aligned}$$

4. (a) The axis of symmetry is horizontal and it is clear from the graph that using rectangles oriented perpendicular to the axis will be easier than using rectangles oriented in parallel. Hence we use the disc-washer method and integrate with respect to x. The outer radius is $R(x) = (2x^2 - x + 2) - 0 = 2x^2 - x + 2$ and the inner radius is $r(x) = x^3 - 0 = x^3$. To determine the upper limit of integration we solve for the points of intersection of the two curves:

$$\begin{aligned} x^3 = 2x^2 - x + 2 \quad &\Longrightarrow \quad x^3 - 2x^2 + x - 2 = 0 \\ &\Longrightarrow \quad x^2(x-2) + (x-2) = (x^2+1)(x-2) = 0 \end{aligned}$$

and so $x = 2$. Hence

$$\begin{aligned} V &= \pi \int_0^2 [(2x^2 - x + 2)^2 - (x^3)^2]\,dx \\ &= \pi \int_0^2 [-x^6 + 4x^4 - 4x^3 + 9x^2 - 4x + 4]\,dx \\ &= \pi \left[-\frac{1}{7}x^7 + \frac{4}{5}x^5 - x^4 + 3x^3 - 2x^2 + 4x \right]_0^2 \\ &= \frac{536\pi}{35}. \end{aligned}$$

(b) The axis of symmetry is horizontal and it is clear from the graph that it will be more easily integrated using rectangles drawn parallel to the axis. Hence we use the shell method and integrate with respect to y; thus we rewrite $y = 2 - x$ as $x = 2 - y$ and observe that of course $y = x$ is the same as the function $x = y$. The average radius is $p(y) = 3 - y$ and the height is $h(y) = (2 - y) - y = 2 - 2y$. To obtain the upper limit of integration, we set

$$2 - y = y \quad \Longrightarrow \quad 2 - 2y = 2(1 - y) = 0$$

and so $y = 1$. So we calculate

$$\begin{aligned} V &= 2\pi \int_{-1}^1 (3 - y)(2 - 2y)\,dy \\ &= 2\pi \int_{-1}^1 (6 - 8y + 2y^2)\,dy \\ &= 2\pi \left[6y - 4y^2 + \frac{2}{3}y^3 \right]_{-1}^1 \\ &= 2\pi \left[\frac{40}{3} \right] \\ &= \frac{80\pi}{3}. \end{aligned}$$

(c) The axis of symmetry is vertical, and from the graph we can deduce that it will be easier to integrate using rectangles oriented perpendicular to the axis. Hence we resort to the disc-washer method and integrate with respect to y. The outer radius is $R(y) = -y - (-2) = 2 - y$ and the inner radius is $r(y) = (y^2 - 2) - (-2) = y^2$. The points of intersection are found by setting

$$y^2 - 2 = -y \quad \Longrightarrow \quad y^2 + y - 2 = (y + 2)(y - 1) = 0$$

giving $y = -2$ and $y = 1$. Thus

$$\begin{aligned}
V &= \pi \int_{-2}^{1} [(2-y)^2 - (y^2)^2]\,dy \\
&= \pi \int_{-2}^{1} [4 - 4y + y^2 - y^4]\,dy \\
&= \pi \left[4y - 2y^2 + \frac{1}{3}y^3 - \frac{1}{5}y^5\right]_{-2}^{1} \\
&= \frac{72\pi}{5}.
\end{aligned}$$