Integration: A Strategy Guide

Now that we have several integration techniques at our disposal, a major challenge involved in evaluating an integral is identifying *which* technique we should use.

Unfortunately, there's no precise answer to this question. (Sorry.) One of the things that makes integration harder than differentiation is that it's not algorithmic. For differentiation, if we see a product, we know that we need to use the Product Rule. For integration, we don't have these kinds of universal rules of thumb.

That being said, however, we can offer some guidelines that will often work. With plenty of practice, you should find yourself developing a better sense of which technique to try first for each integral you encounter. Ultimately, however, you should be aware that integration involves a lot of trial and error (albeit *educated* trial and error), and so you may need to make several attempts before you're able to evaluate a given integral. If at first you don't succeed, try, try again — it might be a cliché, but it's true!

- **Don't make things unnecessarily complicated.** Is the integral an elementary integral? If so, go ahead and evaluate it directly. Remember that even some integrals that seem to be somewhat complicated, such as $\int \sec(x)\tan(x)\,dx$, are just elementary integrals. Similarly, be on the lookout for inverse trig-type integrals. An integral like $\int \frac{1}{\sqrt{1-x^2}}\,dx$ can be integrated using more advanced techniques (such as trig substitution), but why go through all that hassle when you could just recognise that it evaluates as $\arcsin(x) + C$? Review the list of elementary integrals, and make every effort to memorise all of them.

- **Don't make things unnecessarily complicated, even if it seems like they are.** Before we get into more formal techniques for integration, remember that there are a few ways that you can rewrite a given integral that may suddenly bring elementary techniques back into play. For instance, if the integral possesses a sum or difference in the numerator, try splitting it up into two different integrals, as with $\int \frac{x^2+1}{x}\,dx$. If the integral consists of an improper rational function, carry out long division. If there is a trinomial present, completing the square is often useful. If there are trig functions involved, a suitable identity might make the situation much easier.

- **Start simple.** If the given integral is not elementary and can't immediately be written in terms of elementary integrals, begin by considering the most broadly applicable technique we've learned: u-substitution. In particular, if both a certain expression and its derivative are present in the integral, the choice of substitution should be clear. But don't forget that u-substitution can sometimes be useful in less obvious situations, such as $\int (x-2)\sqrt{5-x}\,dx$.

- **Classify the integral.** If nothing has worked so far, it's time to consider one of the other techniques — integration by parts, partial fractions, trig integrals, and trig substitution. Sometimes, more than one of these methods may be appropriate for a given integral. Usually, though, it's fairly easy to distinguish which approach is suitable:
  - Does the integral consist of a (proper) rational function? If so, decompose it into partial fractions.
  - Does the integral consist of a sin/cos pair, a tan/sec pair, or a cot/csc pair, or can it be rewriten in terms of one of these pairs? If so, try a corresponding trig integral strategy.
  - Does the integral possess a factor of the form $\sqrt{k^2+x^2}$, $\sqrt{k^2-x^2}$ or $\sqrt{x^2-k^2}$? If so, try the appropriate trigonometric substitution.
  - If you answered "no" to all of the above, integration by parts is probably the method to use — especially (though not exclusively) if the integral consists of a product. In particular, remember that we're often looking for a product where one factor is easy to integrate, and the other becomes simpler when it's differentiated.
- **Combine approaches.** If nothing has worked so far, try combining these techniques. In particular, it may help to make a suitable u-substitution first, and then try something like integration by parts. Furthermore, remember that trig substitution often leads to an integral which can only be evaluated using one of the trig integral strategies.
- **Get creative. Carefully.** If none of the obvious methods work out, even in combination, you may be faced with an integral that just doesn't fit in any of the usual boxes. For example, perhaps it *looks* like a trig integral strategy is appropriate, but in fact a suitable trigonometric identity will lead to the use of integration by parts. Don't be afraid to try some offbeat approaches — but first, make sure that you haven't missed a more obvious method (or haven't made a mistake with the techniques that you've already tried). Above all, remember that any integral that you'll see on the MATH 1001 final exam *can* be evaluated by the methods we've learned in class. (Honest!)