Advanced Techniques for Integration

Partial Fractions

When integrating a proper rational function $\frac{P(x)}{Q(x)}$ (where $P(x)$ and $Q(x)$ are polynomials) that is not an elementary integral and cannot be evaluated by u-substitution, we can instead decompose the function into partial fractions. The decomposition is determined by the factors of the denominator $Q(x)$ as follows:

- If $ax+b$ is a *distinct* linear factor then $\dfrac{A}{ax+b}$ is a term of the decomposition (for some constant A).
- If ax^2+bx+c is a *distinct* irreducible quadratic factor (that is, a quadratic factor that cannot be factored any further) then $\dfrac{Ax+B}{ax^2+bx+c}$ is a term of the decomposition (for constants A and B).
- If $(ax+b)^n$ is a set of *repeated* linear factors then
$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_n}{(ax+b)^n}$$
are terms of the decomposition (for constants A_1, A_2, $\ldots$, A_n).
- If $(ax^2+bx+c)^n$ is a set of *repeated* irreducible quadratic factors then
$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$$
are terms of the decomposition (for constants A_1, B_1, A_2, B_2, $\ldots$, A_n, B_n).

Trigonometric Integrals

Trigonometric identities can be used to integrate functions that are products of sines and cosines, tangents and secants, or cotangents and cosecants. Most (but not all) of these strategies ultimately lead to u-substitution.

Sine/cosine products

- If the integral involves $\sin^n(x)$ for odd n, factor out a $\sin(x)$ and use the identity $\sin^2(x) = 1-\cos^2(x)$ to convert the remaining $\sin(x)$ factors to $\cos(x)$. Then let $u=\cos(x)$ so $du=-\sin(x)\,dx$.
- If the integral involves $\cos^n(x)$ for odd n, factor out a $\cos(x)$ and use the identity $\cos^2(x) = 1-\sin^2(x)$ to convert the remaining $\cos(x)$ factors to $\sin(x)$. Then let $u=\sin(x)$ so $du=\cos(x)\,dx$.
- If the power of neither $\sin(x)$ nor $\cos(x)$ is odd, rewrite the integral using the half-angle formulas
$$\sin^2(\theta)=\frac{1-\cos(2\theta)}{2} \quad \text{and} \quad \cos^2(\theta)=\frac{1+\cos(2\theta)}{2}.$$

Tangent/secant products

- If the integral involves $\tan^n(x)$ for odd n, factor out a $\sec(x)\tan(x)$ (if possible) and use the identity $\tan^2(x) = \sec^2(x) - 1$ to convert the remaining $\tan(x)$ factors to $\sec(x)$. Then let $u = \sec(x)$ so $du = \sec(x)\tan(x)\,dx$.
  - If there are no factors of $\sec(x)$ present, instead try factoring out a $\tan^2(x)$ and apply the identity $\tan^2(x) = \sec^2(x) - 1$ to it.
- If the integral involves $\sec^n(x)$ for even n, factor out a $\sec^2(x)$ and use the identity $\sec^2(x) = \tan^2(x) + 1$ to convert the remaining $\sec(x)$ factors to $\tan(x)$. Then let $u = \tan(x)$ so $du = \sec^2(x)\,dx$.
- Otherwise, try integration by parts with $dv = \sec^2(x)\,dx$.

Cotangent/cosecant products

The strategies for tangent/secant products apply in exactly the same way to products involving cotangent and cosecant functions, using the identity $\cot^2(x) + 1 = \csc^2(x)$.

Any trigonometric products

If none of the preceding strategies work, try rewriting the integral in terms of different functions — for example, write $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ or $\sec(x) = \dfrac{1}{\cos(x)}$ — and then try the strategies which apply to that new trigonometric product.

Trigonometric Substitution

If the integral involves certain radical forms and cannot be evaluated as an elementary integral or via u-substitution, the integrand can be simplified by replacing x with an appropriate trigonometric function as follows:

- If the integral involves $\sqrt{x^2 + k^2}$, let $x = k\tan(\theta)$ so $dx = k\sec^2(\theta)\,d\theta$, and simplify using the identity $\tan^2(\theta) + 1 = \sec^2(\theta)$.
- If the integral involves $\sqrt{k^2 - x^2}$, let $x = k\sin(\theta)$ so $dx = k\cos(\theta)\,d\theta$, and simplify using the identity $1 - \sin^2(\theta) = \cos^2(\theta)$.
- If the integral involves $\sqrt{x^2 - k^2}$, let $x = k\sec(\theta)$ so $dx = k\sec(\theta)\tan(\theta)\,d\theta$, and simplify using the identity $\sec^2(\theta) - 1 = \tan^2(\theta)$.
- If x in these three expressions is instead replaced by a function $f(x)$, perform u-substitution with $u = f(x)$ before using the appropriate trigonometric substitution.
- If there is a trinomial under the square root, try completing the square and then using u-substitution before applying trigonometric substitution.