1. We will use a regular partition of $[-1, 1/2]$ with

$$\Delta x = \frac{1/2 - (-1)}{n} = \frac{3}{2n}$$

$$x_i^* = x_i = -1 + \frac{3i}{2n}$$

$$f(x_i^*) = 9 - 4\left(-1 + \frac{3i}{2n}\right)^2$$

$$= 9 - 4\left(1 - \frac{3i}{n} + \frac{9i^2}{4n^2}\right)$$

$$= 5 + \frac{12i}{n} - \frac{9i^2}{n^2}$$

Now we can write

$$\int_{-1}^{1/2} (9 - 4x^2)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(5 + \frac{12i}{n} - \frac{9i^2}{n^2}\right) \cdot \frac{3}{2n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{15}{2n} + \frac{18i}{n^2} - \frac{27i^2}{2n^3}\right)$$

$$= \lim_{n\to\infty} \left[\frac{15}{2n} \sum_{i=1}^{n} 1 + \frac{18}{n^2} \sum_{i=1}^{n} i - \frac{27}{2n^3} \sum_{i=1}^{n} i^2\right]$$

$$= \lim_{n\to\infty} \left[\frac{15}{2n} \cdot n + \frac{18}{n^2} \cdot \frac{n(n+1)}{2} - \frac{27}{2n^3} \cdot \frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty} \left[\frac{15}{2} + \frac{9(n+1)}{n} - \frac{9(n+1)(2n+1)}{4n^2}\right]$$

$$= \frac{15}{2} + 9 - \frac{9}{2}$$

$$\boxed{= 12}$$

2. a) $\displaystyle\int_1^4 \frac{1}{\sqrt{x}\,(1+3\sqrt{x})^2}\,dx$

We let $u = 1+3\sqrt{x}$

$du = 3\cdot\frac{1}{2}x^{-1/2}\,dx$

$\frac{2}{3}\,du = \frac{1}{\sqrt{x}}\,dx$

When $x=1$, $u = 1+3\sqrt{1} = 4$

$x=4$, $u = 1+3\sqrt{4} = 7$

The integral becomes

$$\int_1^4 \frac{1}{\sqrt{x}(1+3\sqrt{x})^2}\,dx = \frac{2}{3}\int_4^7 u^{-2}\,du$$

$$= \frac{2}{3}\left[\frac{u^{-1}}{-1}\right]_4^7$$

$$= -\frac{2}{3}\left[\frac{1}{7}-\frac{1}{4}\right]$$

$$= -\frac{2}{3}\left[\frac{4}{28}-\frac{7}{28}\right]$$

$$\boxed{= \frac{1}{14}}$$

2. b) $\int \frac{20e^{2x}}{25+16e^{4x}}\,dx$

Observe that $\frac{20e^{2x}}{25+16e^{4x}} = \frac{20e^{2x}}{25+16(e^{2x})^2}$

We let $u = e^{2x}$ so $du = 2e^{2x}\,dx$

$\frac{1}{2}du = e^{2x}\,dx$

The integral becomes

$$\int \frac{20e^{2x}}{25+16e^{4x}}\,dx = \frac{1}{2}\int \frac{20}{25+16u^2}\,du$$

$$= 10\int \frac{1}{25+16u^2}\,du$$

$$= \frac{10}{16}\int \frac{1}{u^2+\left(\frac{5}{4}\right)^2}\,du$$

$$= \frac{5}{8}\cdot\frac{1}{5/4}\arctan\left(\frac{u}{5/4}\right)+C$$

$$\boxed{= \frac{1}{2}\arctan\left(\frac{4}{5}e^{2x}\right)+C}$$

2. c) $\int \frac{6x^5}{x^3+4}\,dx$

We use long division:

$$\begin{array}{r} 6x^2 \\ x^3+4 \overline{\big) \; 6x^5 \qquad\quad} \\ \underline{6x^5 + 24x^2} \\ -24x^2 \end{array}$$

So we can write

$$\int \frac{6x^5}{x^3+4}\,dx = \int \left(6x^2 + \frac{-24x^2}{x^3+4}\right) dx$$

$$= 6\int x^2\,dx - 24\int \frac{x^2}{x^3+4}\,dx$$

Let $u = x^3+4$

$du = 3x^2\,dx$

$\frac{1}{3}\,du = x^2\,dx$

$$= 6 \cdot \frac{x^3}{3} - 24\int \frac{x^2}{x^3+4}\,dx$$

$$= 2x^3 - 24 \cdot \frac{1}{3}\int \frac{1}{u}\,du$$

$$= 2x^3 - 8\ln|u| + C$$

$$\boxed{= 2x^3 - 8\ln|x^3+4| + C}$$

2. d) $\int e^{7x}\sin(x)\,dx$

We use integration by parts with

$w = e^{7x}$ $\quad$ $dw = 7e^{7x}\,dx$

$dv = \sin(x)\,dx$ $\quad$ $v = -\cos(x)$

so $$\int e^{7x}\sin(x)\,dx = -e^{7x}\cos(x) + 7\int e^{7x}\cos(x)\,dx$$

We use integration by parts again, now with

$w = e^{7x}$ $\quad$ $dw = 7e^{7x}\,dx$

$dv = \cos(x)\,dx$ $\quad$ $v = \sin(x)$

so $$\int e^{7x}\sin(x)\,dx = -e^{7x}\cos(x) + 7\left[e^{7x}\sin(x) - 7\int e^{7x}\sin(x)\,dx\right]$$

$$= -e^{7x}\cos(x) + 7e^{7x}\sin(x) - 49\int e^{7x}\sin(x)\,dx$$

$$50\int e^{7x}\sin(x)\,dx = -e^{7x}\cos(x) + 7e^{7x}\sin(x)$$

$$\boxed{\int e^{7x}\sin(x)\,dx = -\frac{1}{50}e^{7x}\cos(x) + \frac{7}{50}e^{7x}\sin(x) + C}$$

3. a) $\displaystyle\int \frac{\sin^5\left(\frac{x}{3}\right)}{\sqrt{\cos\left(\frac{x}{3}\right)}}\,dx$

$$= \int \frac{\sin^4\left(\frac{x}{3}\right)}{\sqrt{\cos\left(\frac{x}{3}\right)}} \cdot \sin\left(\frac{x}{3}\right) dx$$

$$= \int \frac{\left[1-\cos^2\left(\frac{x}{3}\right)\right]^2}{\sqrt{\cos\left(\frac{x}{3}\right)}} \cdot \sin\left(\frac{x}{3}\right) dx$$

Let $u = \cos\left(\frac{x}{3}\right)$

$du = -\frac{1}{3}\sin\left(\frac{x}{3}\right)dx$

$-3\,du = \sin\left(\frac{x}{3}\right)dx$

$$= -3\int \frac{\left[1-u^2\right]^2}{\sqrt{u}}\,du$$

$$= -3\int \frac{1-2u^2+u^4}{\sqrt{u}}\,du$$

$$= -3\int \left(u^{-1/2} - 2u^{3/2} + u^{7/2}\right) du$$

$$= -3\left[\frac{u^{1/2}}{1/2} - 2\cdot\frac{u^{5/2}}{5/2} + \frac{u^{9/2}}{9/2}\right] + C$$

$$\boxed{= -6\sqrt{\cos\left(\frac{x}{3}\right)} + \frac{12}{5}\sqrt{\cos^5\left(\frac{x}{3}\right)} - \frac{2}{3}\sqrt{\cos^9\left(\frac{x}{3}\right)} + C}$$

3. b) $\int \frac{\sqrt{x^2-9}}{x^3}\,dx$

Let $x = 3\sec(\theta)$ so $dx = 3\sec(\theta)\tan(\theta)\,d\theta$ and

$$\sqrt{x^2-9} = \sqrt{9\sec^2(\theta)-9} = \sqrt{9\tan^2(\theta)} = 3\tan(\theta)$$

so we have

$$\int \frac{\sqrt{x^2-9}}{x^3}\,dx = \int \frac{3\tan(\theta)}{27\sec^3(\theta)} \cdot 3\sec(\theta)\tan(\theta)\,d\theta$$

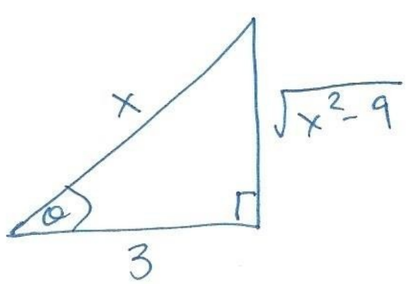
$$= \frac{1}{3}\int \frac{\tan^2(\theta)}{\sec^2(\theta)}\,d\theta$$

$$= \frac{1}{3}\int \frac{\sin^2(\theta)}{\cos^2(\theta)} \cdot \cos^2(\theta)\,d\theta$$

$$= \frac{1}{3}\int \sin^2(\theta)\,d\theta$$

$$= \frac{1}{3}\int \frac{1-\cos(2\theta)}{2}\,d\theta$$

$$= \frac{1}{6}\int \left[1-\cos(2\theta)\right]d\theta$$



$$= \frac{1}{6}\left[\theta - \frac{1}{2}\sin(2\theta)\right] + C$$

$$= \frac{1}{6}\left[\theta - \sin(\theta)\cos(\theta)\right] + C$$

$$= \frac{1}{6}\left[\operatorname{arcsec}\left(\frac{x}{3}\right) - \frac{\sqrt{x^2-9}}{x} \cdot \frac{3}{x}\right] + C$$

$$\boxed{= \frac{1}{6}\operatorname{arcsec}\left(\frac{1}{3}x\right) - \frac{\sqrt{x^2-9}}{2x^2} + C}$$

3. c) $\displaystyle\int \frac{7}{25x^2-20x+13}\,dx$

Can we factor the denominator?

$$25x^2-20x+13=0$$

$$x=\frac{20\pm\sqrt{400-4\cdot 25\cdot 13}}{50}=\frac{20\pm\sqrt{-900}}{50}$$

Since the polynomial has no real roots, it can't be factored. Thus we try to complete the square:

$$25x^2-20x+13=25\left[x^2-\frac{4}{5}x+\frac{13}{25}\right]$$

$$=25\left[\left(x^2-\frac{4}{5}x+\frac{4}{25}\right)+\frac{13}{25}-\frac{4}{25}\right]$$

$$=25\left[\left(x-\frac{2}{5}\right)^2+\frac{9}{25}\right]$$

$$=5^2\left(x-\frac{2}{5}\right)^2+9$$

$$=(5x-2)^2+9$$

The integral becomes

$$\int\frac{7}{25x^2-20x+13}\,dx=\int\frac{7}{(5x-2)^2+9}\,dx$$

Let $u=5x-2$

$du=5\,dx$

$\frac{1}{5}\,du=dx$

$$=\frac{7}{5}\int\frac{1}{u^2+9}\,du$$

$$=\frac{7}{5}\cdot\frac{1}{3}\arctan\left(\frac{u}{3}\right)+C$$

$$\boxed{=\frac{7}{15}\arctan\left(\frac{5x-2}{3}\right)+C}$$

3. d) $\int \frac{5x^2}{(x+2)(x^2+1)}\,dx$

The form of the partial fraction decomposition is

$$\frac{5x^2}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$$

$$5x^2 = A(x^2+1) + (Bx+C)(x+2)$$

For $x=-2$: $20 = A\cdot 5 + 0 \rightarrow A=4$

For $x=0$: $0 = A\cdot 1 + (0+C)\cdot 2$

$0 = 4+2C \rightarrow C=-2$

For $x=1$: $5 = A\cdot 2 + (B\cdot 1 + C)\cdot 3$

$5 = 8 + 3B - 6$

$3 = 3B \rightarrow B=1$

The integral becomes

$$\int \left(\frac{4}{x+2} + \frac{x-2}{x^2+1}\right)dx$$

$$= 4\int \frac{1}{x+2}\,dx + \int \frac{x}{x^2+1}\,dx - 2\int \frac{1}{x^2+1}\,dx$$

$$= 4\ln|x+2| - 2\arctan(x) + \int \frac{x}{x^2+1}\,dx$$

Let $u = x^2+1$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

$$= 4\ln|x+2| - 2\arctan(x) + \frac{1}{2}\int \frac{1}{u}\,du$$

$$= 4\ln|x+2| - 2\arctan(x) + \frac{1}{2}\ln|u| + C$$

$$= 4\ln|x+2| - 2\arctan(x) + \frac{1}{2}\ln|x^2+1| + C$$

$$\boxed{= 4\ln|x+2| - 2\arctan(x) + \frac{1}{2}\ln(x^2+1) + C}$$

3. e) $\int_{-\infty}^{0} xe^x \, dx = \lim_{T \to -\infty} \int_T^0 xe^x \, dx$

We use integration by parts with

$w = x \qquad dw = dx$

$dv = e^x \, dx \qquad v = e^x$

so $\int_{-\infty}^{0} xe^x \, dx = \lim_{T \to -\infty} \left(\left[xe^x \right]_T^0 - \int_T^0 e^x \, dx \right)$

$= \lim_{T \to -\infty} \left[xe^x - e^x \right]_T^0$

$= \lim_{T \to -\infty} \left[(0 - 1) - (Te^T - e^T) \right]$

$= \lim_{T \to -\infty} (-1 - Te^T + e^T)$

$= -1 - \lim_{T \to -\infty} Te^T \qquad (0 \cdot \infty \text{ form})$

$= -1 - \lim_{T \to -\infty} \frac{T}{e^{-T}} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$

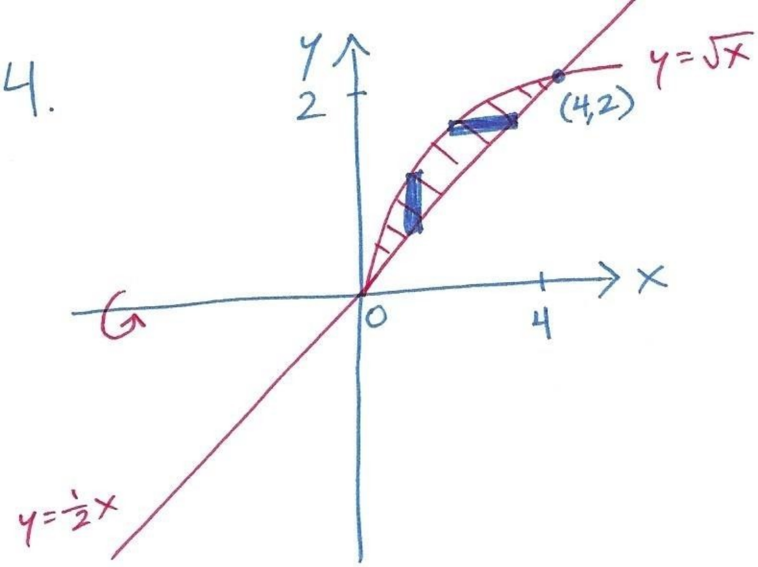
$\overset{H}{=} -1 - \lim_{T \to -\infty} \frac{1}{-e^{-T}}$

$= -1 + \lim_{T \to -\infty} e^T$

$= -1 + 0$

$\boxed{= -1}$

4.



We set $\frac{1}{2}x=\sqrt{x}$

$$\left(\tfrac{1}{2}x\right)^2 = x$$

$$\tfrac{1}{4}x^2 - x = 0$$

$$x\left(\tfrac{1}{4}x-1\right)=0$$

$$x=0 \quad \text{or} \quad \tfrac{1}{4}x-1=0$$

$$x=4$$

a) $A = \int_0^4 \left(\sqrt{x} - \frac{1}{2}x\right) dx$

b) As a function of y, $y=\sqrt{x}$ becomes $x=y^2$

$y=\frac{1}{2}x$ becomes $x=2y$

$$A = \int_0^2 (2y - y^2)\,dy$$

c) $R(x) = \sqrt{x}$

$r(x) = \frac{1}{2}x$

$$V = \pi\int_0^4 \left(\left[\sqrt{x}\right]^2 - \left[\tfrac{1}{2}x\right]^2\right) dx$$

$$= \pi\int_0^4 \left(x - \tfrac{1}{4}x^2\right) dx$$

5. $f(x) = \int_{\sqrt{x}}^{3} \frac{6t^5}{\sqrt{4t^4 - 9}}\, dt$

$= -\int_{3}^{\sqrt{x}} \frac{6t^5}{\sqrt{4t^4 - 9}}\, dt$

$f'(x) = -\frac{6(\sqrt{x})^5}{\sqrt{4(\sqrt{x})^4 - 9}} \cdot [\sqrt{x}]'$

$= -\frac{6x^{5/2}}{\sqrt{4x^2 - 9}} \cdot \frac{1}{2\sqrt{x}}$

$\boxed{= -\frac{3x^2}{\sqrt{4x^2 - 9}}}$

6. a) $ty\frac{dy}{dt} = t^3 - 1$

$$y\,dy = \frac{t^3-1}{t}\,dt$$

$$\int y\,dy = \int \left(t^2 - \frac{1}{t}\right)dt$$

$$\frac{1}{2}y^2 = \frac{1}{3}t^3 - \ln|t| + C \qquad \text{(general solution)}$$

We are given that $y=4$ when $t=1$, so

$$8 = \frac{1}{3} - 0 + C$$

$$C = \frac{23}{3}$$

The particular solution is given by

$$\frac{1}{2}y^2 = \frac{1}{3}t^3 - \ln|t| + \frac{23}{3}$$

$$y^2 = \frac{2}{3}t^3 - 2\ln|t| + \frac{46}{3}$$

$$\boxed{y = \pm\sqrt{\frac{2}{3}t^3 - 2\ln|t| + \frac{46}{3}}}$$

6. b) Let $y(t)$ be the number of millions of bacteria in the culture after t weeks. We are given that $y(2) = 10$ and $y(6) = 40$. We know that

$$y(t) = y_0 e^{kt}.$$

Then
$$y(2) = y_0 e^{2k} = 10$$
$$y(6) = y_0 e^{6k} = 40$$

We have
$$\frac{y_0 e^{6k}}{y_0 e^{2k}} = \frac{40}{10}$$
$$e^{6k-2k} = 4$$
$$e^{4k} = 4$$
$$4k = \ln(4) \rightarrow k = \frac{\ln(4)}{4}$$

This means that
$$y_0 e^{2 \cdot \frac{\ln(4)}{4}} = 10$$
$$y_0 e^{\frac{1}{2}\ln(4)} = 10$$
$$y_0 e^{\ln(2)} = 10$$
$$y_0 \cdot 2 = 10 \rightarrow y_0 = 5$$

The initial culture consisted of 5 million bacteria.

6. c) First we note that $f(x) \geq 0$ for all x.

Next we compute

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^3 6\,(x+3)^{-2}\,dx$$

$$= 6\left[\frac{(x+3)^{-1}}{-1}\right]_0^3$$

$$= 6\left[-\frac{1}{6}+\frac{1}{3}\right]$$

$$= 6 \cdot \frac{1}{6}$$

$$= 1$$

Since $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$, $f(x)$ is a probability density function.

Next, $P(0 \leq X \leq 1) = \int_0^1 f(x)\,dx$

$$= \int_0^1 6\,(x+3)^{-2}\,dx$$

$$= 6\left[\frac{(x+3)^{-1}}{-1}\right]_0^1$$

$$= 6\left[-\frac{1}{4}+\frac{1}{3}\right]$$

$$= 6 \cdot \frac{1}{12}$$

$$\boxed{= \frac{1}{2}}$$

7. We use integration by parts with

$$w = \sec^{n-2}(x) \qquad dw = (n-2)\sec^{n-3}(x)\cdot \sec(x)\tan(x)\,dx$$

$$= (n-2)\sec^{n-2}(x)\tan(x)\,dx$$

$$dv = \sec^2(x)\,dx \qquad v = \tan(x)$$

So we have

$$\int \sec^n(x)\,dx = \tan(x)\sec^{n-2}(x) - (n-2)\int \sec^{n-2}(x)\tan^2(x)\,dx$$

$$= \tan(x)\sec^{n-2}(x) - (n-2)\int \sec^{n-2}(x)\left[\sec^2(x)-1\right]dx$$

$$= \tan(x)\sec^{n-2}(x) - (n-2)\int\left[\sec^n(x) - \sec^{n-2}(x)\right]dx$$

$$= \tan(x)\sec^{n-2}(x) - (n-2)\int \sec^n(x)\,dx + (n-2)\int \sec^{n-2}(x)\,dx$$

$$(n-1)\int \sec^n(x)\,dx = \tan(x)\sec^{n-2}(x) + (n-2)\int \sec^{n-2}(x)\,dx$$

$$\int \sec^n(x)\,dx = \frac{\tan(x)\sec^{n-2}(x)}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx$$

Now we can write

$$\int \sec^5(x)\,dx = \frac{\tan(x)\sec^3(x)}{4} + \frac{3}{4}\int \sec^3(x)\,dx$$

$$= \frac{1}{4}\tan(x)\sec^3(x) + \frac{3}{4}\left[\frac{\tan(x)\sec(x)}{2} + \frac{1}{2}\int \sec(x)\,dx\right]$$

$$\boxed{= \frac{1}{4}\tan(x)\sec^3(x) + \frac{3}{8}\tan(x)\sec(x) + \frac{3}{8}\ln\left|\sec(x)+\tan(x)\right| + C}$$