MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 4.4 **Math 1000 Worksheet** FALL 2023

SOLUTIONS

1. (a) First we determine the critical points of $f(x)$. We have

$$f'(x) = 3x^2 - 9 = 3(x^2 - 3)$$

so $f'(x) = 0$ when $x = \pm\sqrt{3}$ and $f'(x)$ is never undefined. Hence $x = \pm\sqrt{3}$ are the only critical points. Next we evaluate $f(x)$ at the critical points and at the endpoints $x = -4$ and $x = 3$:

$$f\left(-\sqrt{3}\right) = 6\sqrt{3} \approx 10.4, \quad f\left(\sqrt{3}\right) = -6\sqrt{3} \approx -10.4$$

$$f(-4) = -28, \quad f(3) = 0.$$

Hence the maximum value of $f(x)$ on $-4 \leq x \leq 3$ is $6\sqrt{3}$ and the minimum value is -28.

(b) First we identify the critical points. Note that

$$f'(x) = \frac{(2x)(x+1) - (1)(x^2+3)}{(x+1)^2} = \frac{(x+3)(x-1)}{(x+1)^2},$$

which is zero for $x = 1$ and $x = -3$, and fails to exist at $x = -1$. For the interval $0 \leq x \leq 4$, then, the only critical point is $x = 1$, for which $f(1) = 2$. Checking the endpoints, we have $f(0) = 3$ and $f(4) = \frac{19}{5} = 3.8$. Hence the maximum value of $f(x)$ is $\frac{19}{5}$, and the minimum value is 2.

(c) We again begin by determining the critical points of $f(x)$, observing that

$$f'(x) = \sec(x)\tan(x).$$

Setting $f'(x) = 0$ gives $x = 0$, and $f'(x)$ never fails to exist on the given interval (though it does fail to exist for many other values of x). Hence $x = 0$ is the only critical point. We evaluate $f(x)$ there and at the endpoints, giving

$$f(0) = 1, \quad f\left(-\frac{\pi}{6}\right) = \frac{2\sqrt{3}}{3} \approx 1.15, \quad f\left(\frac{\pi}{3}\right) = 2.$$

Thus the maximum value of $f(x)$ on the given interval is 2 and the minimum value is 1.

(d) As before, we find the critical points. Differentiation gives

$$f'(x) = 1 + 2\sin(x)$$

which fails to exist nowhere, and is zero for $x = -\frac{\pi}{6}$ and $x = -\frac{5\pi}{6}$, both of which are on the interval $-\pi \le x \le \pi$. Note that

$$f\left(-\frac{\pi}{6}\right) = -\frac{\pi}{6} - 2\cos\left(-\frac{\pi}{6}\right) \approx -2.26$$

and

$$f\left(-\frac{5\pi}{6}\right) = -\frac{5\pi}{6} - 2\cos\left(-\frac{5\pi}{6}\right) \approx -0.89.$$

At the endpoints,

$$f(-\pi) = -\pi - 2\cos(-\pi) \approx -1.14 \quad \text{and} \quad f(\pi) = \pi - 2\cos(\pi) \approx 5.14.$$

Hence the maximum value of $f(x)$ on $-\pi \le x \le \pi$ is approximately $\boxed{5.14}$, while the minimum value is about $\boxed{-2.26}$.

2. (a) Observe that

$$f'(x) = 12x - 6x^2 = 6x(2 - x).$$

This is never undefined, and equals zero for $x = 0$ and $x = 2$, of which only the latter lies on the indicated interval. Since the requirements of the Second Derivative Test are satisfied, we find

$$f''(x) = 12 - 12x \quad \Longrightarrow \quad f''(2) = -12 < 0.$$

Thus $x = 2$ is the absolute maximum, and the maximum value of $f(x)$ on the interval $1 < x < 7$ is $\boxed{f(2) = 10}$.

(b) From part (a), we know that $x = 0$ is the only critical point on the interval $-7 < x < 1$, and $f''(0) = 12 > 0$. Thus $x = 0$ is the absolute minimum, and the minimum value of $f(x)$ on this interval is $\boxed{f(0) = 2}$.

3. We have

$$f'(x) = \frac{12 + 16x - 3x^2}{(3x - 8)^2}.$$

This is undefined for $x = \frac{8}{3}$, which is not on the given interval. It is zero when $x = 6$ and $x = -\frac{2}{3}$, of which only the latter is on $-2 < x < 2$. Since we have only one critical point, we can use the Second Derivative Test. Observe that

$$f''(x) = -\frac{200}{(3x - 8)^3} \quad \Longrightarrow \quad f''\left(-\frac{2}{3}\right) = \frac{1}{5} > 0.$$

Thus $x = -\frac{2}{3}$ is the absolute minimum, and the minimum value of $f(x)$ on this interval is $\boxed{f\left(-\frac{2}{3}\right) = \frac{1}{5}}$.