MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Test 2 — **MATHEMATICS 1000-002** — Fall 2023

SOLUTIONS

[1] 1. (a) Yes, $f(x)$ is continuous at $x = -4$.

[2] (b) No, $f(x)$ is not differentiable at $x = -4$; there is a cusp (or a "sharp corner") at this point.

[2] (c) No, $f(x)$ is not continuous at $x = 0$; it is a non-removable discontinuity.

[2] (d) No, $f(x)$ is not differentiable at $x = 0$; a function cannot be differentiable at a point if it is not continuous at that point.

[1] (e) Yes, $f(x)$ is continuous at $x = 1$.

[1] (f) Yes, $f(x)$ is differentiable at $x = 1$.

[2] (g) No, $f(x)$ is not continuous at $x = 3$; it is a removable discontinuity.

[1] (h) No, $f(x)$ is not differentiable at $x = 3$; a function cannot be differentiable at a point if it is not continuous at that point.

[5] 2. Observe that $f(x)$ is a rational function, so we need only consider one of the limits at infinity. Then

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{(3x^2+1)^2}{x^3(5-6x)} \\
&= \lim_{x\to\infty} \frac{9x^4+6x^2+1}{5x^3-6x^4} \cdot \frac{\frac{1}{x^4}}{\frac{1}{x^4}} \\
&= \lim_{x\to\infty} \frac{9+\frac{6}{x^2}+\frac{1}{x^4}}{\frac{5}{x}-6} \\
&= \frac{9+0+0}{0-6} \\
&= \frac{9}{-6} \\
&= -\frac{3}{2}.
\end{aligned}$$

Hence the only horizontal asymptote is the line $y = -\frac{3}{2}$.

[7] 3. (a) We have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{[(x+h)^2-6(x+h)+9]-[x^2-6x+9]}{h} \\
&= \lim_{h\to 0}\frac{x^2+2xh+h^2-6x-6h+9-x^2+6x-9}{h} \\
&= \lim_{h\to 0}\frac{2xh+h^2-6h}{h} \\
&= \lim_{h\to 0}(2x+h-6) \\
&= 2x-6.
\end{aligned}$$

[3] (b) From part (a), $m = f'(1) = -4$. Furthermore, $y = f(1) = 4$. Thus the equation of the tangent line is

$$y-4=-4(x-1) \quad\Longrightarrow\quad y=-4x+8.$$

[5] 4. (a) First, $f(0) = 3\cdot 0+8 = 8$, which is defined. Next we must check the one-sided limits to determine if $\lim_{x\to 0} f(x)$ exists. We have

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\frac{x+2}{x-1} = \frac{2}{-1} = -2$$

and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\frac{x^2-4x+4}{x-2} = \frac{4}{-2} = -2.$$

Since the one-sided limits agree,

$$\lim_{x\to 0} f(x) = -2$$

and so the limit exists. However, $f(0) \neq \lim_{x\to 0} f(x)$ so $f(x)$ is discontinuous at $x = 0$. Because the limit exists, the discontinuity is removable.

[8] (b) Since part (a) has explored the only point at which the definition of $f(x)$ changes, we must consider any points which would make each part of the piecewise definition undefined.
First, then, we set $x^2-1 = 0$ so $x^2 = 1$ and $x = \pm 1$. However, because the first definition applies only for $x < 0$, we can neglect $x = 1$. Because $f(-1) = \frac{1}{0}$, we know that $\lim_{x\to 1} f(x)$ does not exist and hence $x = -1$ is a non-removable discontinuity.
The second part of the definition, $f(x) = 3x+8$, is a polynomial and hence is always defined.
Finally, we set $x-2 = 0$ so $x = 2$. Because $f(2) = \frac{0}{0}$, we use the Cancellation Method:

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}\frac{x^2-4x+4}{x-2} = \lim_{x\to 2}\frac{(x-2)^2}{x-2} = \lim_{x\to 2}(x-2) = 0.$$

Since the limit exists, $x = 2$ is a removable discontinuity.