MEMORIAL UNIVERSITY OF NEWFOUNDLAND
DEPARTMENT OF MATHEMATICS AND STATISTICS

SOLUTIONS

1. (a) $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{3(x+h)}{x+h-4} - \frac{3x}{x-4}}{h}$$
$$= \lim_{h\to 0} \frac{3(x+h)(x-4) - 3x(x+h-4)}{h(x-4)(x+h-4)} = \lim_{h\to 0} \frac{-12h}{h(x-4)(x+h-4)}$$
$$= \lim_{h\to 0} \frac{-12}{(x-4)(x+h-4)} = -\frac{12}{(x-4)^2}$$

(b) $$f'(t) = \lim_{h\to 0} \frac{f(t+h)-f(t)}{h} = \lim_{h\to 0} \frac{\frac{4}{(t+h)^2+1} - \frac{4}{t^2+1}}{h}$$
$$= \lim_{h\to 0} \frac{4(t^2+1) - 4[(t+h)^2+1]}{h(t^2+1)[(t+h)^2+1]} = \lim_{h\to 0} \frac{-8th-4h^2}{h(t^2+1)[(t+h)^2+1]}$$
$$= \lim_{h\to 0} \frac{-4h(h+2t)}{h(t^2+1)[(t+h)^2+1]} = \lim_{h\to 0} \frac{-4(h+2t)}{(t^2+1)[(t+h)^2+1]} = -\frac{8t}{(t^2+1)^2}$$

(c) $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{2-3(x+h)} - \sqrt{2-3x}}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{2-3(x+h)} - \sqrt{2-3x}}{h} \cdot \frac{\sqrt{2-3(x+h)} + \sqrt{2-3x}}{\sqrt{2-3(x+h)} + \sqrt{2-3x}}$$
$$= \lim_{h\to 0} \frac{[2-3(x+h)] - [2-3x]}{h[\sqrt{2-3(x+h)} + \sqrt{2-3x}]}$$
$$= \lim_{h\to 0} \frac{-3h}{h[\sqrt{2-3(x+h)} + \sqrt{2-3x}]} = \lim_{h\to 0} \frac{-3}{\sqrt{2-3(x+h)} + \sqrt{2-3x}}$$
$$= -\frac{3}{2\sqrt{2-3x}}$$

(d) $$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{\sqrt{(x+h)^2+1} - \sqrt{x^2+1}}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{(x+h)^2+1} - \sqrt{x^2+1}}{h} \cdot \frac{\sqrt{(x+h)^2+1} + \sqrt{x^2+1}}{\sqrt{(x+h)^2+1} + \sqrt{x^2+1}}$$
$$= \lim_{h\to 0} \frac{[(x+h)^2+1] - [x^2+1]}{h\left(\sqrt{(x+h)^2+1} + \sqrt{x^2+1}\right)}$$
$$= \lim_{h\to 0} \frac{2xh+h^2}{h\left(\sqrt{(x+h)^2+1} + \sqrt{x^2+1}\right)} = \lim_{h\to 0} \frac{2x+h}{\sqrt{(x+h)^2+1} + \sqrt{x^2+1}}$$
$$= \frac{2x}{2\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}$$

2. The derivative of $f(x)$ is

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{[(x+h)^3 - 3(x+h)^2 + 2(x+h)] - [x^3 - 3x^2 + 2x]}{h} \\ &= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - 6xh - 3h^2 + 2h}{h} \\ &= \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2 - 6x - 3h + 2)}{h} \\ &= \lim_{h\to 0} (3x^2 + 3xh + h^2 - 6x - 3h + 2) \\ &= 3x^2 - 6x + 2. \end{aligned}$$

At $x = 3$, $f'(3) = 11$ so the equation of the tangent line has the form $y = 11x + b$. Since $f(3) = 6$, we know that

$$6 = 11(3) + b \quad \Longrightarrow \quad b = -27$$

and so the equation of the tangent line is $y = 11x - 27.$ Finally, note that $f'(-1) = 11$ as well, so since the tangent lines to $f(x)$ at $x = 3$ and at $x = -1$ have the same slope, they must be parallel.

3. First note that

$$|3x+6| = \begin{cases} 3x+6 & \text{for } x \geq -2 \\ -(3x+6) & \text{for } x < -2. \end{cases}$$

Using the alternative definition of the derivative,

$$\lim_{x\to -2^-} \frac{f(x)-f(-2)}{x+2} = \lim_{x\to -2^-} \frac{-3x-6-0}{x+2} = \lim_{x\to -2^-} \frac{-3(x+2)}{x+2} = \lim_{x\to -2^-} (-3) = -3,$$

whereas

$$\lim_{x\to -2^+} \frac{f(x)-f(-2)}{x+2} = \lim_{x\to -2^+} \frac{(3x+6)-0}{x+2} = \lim_{x\to -2^+} \frac{3(x+2)}{x+2} = \lim_{x\to -2^+} 3 = 3.$$

Since the one-sided limits are not equal,

$$f'(-2) = \lim_{x\to -2} \frac{f(x)-f(-2)}{x+2}$$

does not exist, and hence $f(x)$ is non-differentiable at $x = -2$.

4. By the alternative definition of the derivative,

$$f'(-4) = \lim_{x\to -4} \frac{f(x)-f(-4)}{x-(-4)} = \lim_{x\to -4} \frac{f(x)+31}{x+4}.$$

From the left,

$$\lim_{x\to -4^-} \frac{f(x)+31}{x+4} = \lim_{x\to -4^-} \frac{(17-3x^2)+31}{x+4} = \lim_{x\to -4^-} \frac{48-3x^2}{x+4}$$
$$= \lim_{x\to -4^-} \frac{-3(x-4)(x+4)}{x+4} = \lim_{x\to -4^-} -3(x-4) = 24.$$

From the right,

$$\lim_{x\to -4^+} \frac{f(x)+31}{x+4} = \lim_{x\to -4^+} \frac{(x^3+3x^2-15)+31}{x+4} = \lim_{x\to -4^+} \frac{x^3+3x^2+16}{x+4}$$
$$= \lim_{x\to -4^+} \frac{(x+4)(x^2-x+4)}{x+4} = \lim_{x\to -4^+} (x^2-x+4) = 24.$$

Since the one-sided limits agree, the limit exists and therefore $f(x)$ is differentiable at $x = -4$.