MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Section 1.6 **Math 1000 Worksheet** Fall 2023

SOLUTIONS

1. (a) First, we observe that

$$f(-2) = \frac{(-2)^2 + 4}{2(-2)^2 + 4} = \frac{8}{12} = \frac{2}{3},$$

so $f(-2)$ is defined. Next, we evaluate

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} \frac{x^2 + 4}{2x^2 + 4} = \frac{2}{3}$$

and

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} \frac{x^2 - 4}{2x + 4} = \lim_{x \to -2^+} \frac{(x-2)(x+2)}{2(x+2)} = \lim_{x \to -2^+} \frac{x-2}{2} = -2.$$

Since the one-sided limits are not equal, $\lim_{x \to -2} f(x)$ does not exist. Hence $f(x)$ is not continuous at $x = -2$, and it is a non-removable discontinuity.

(b) First, we observe that

$$f(1) = \frac{1^2 - 4}{2 \cdot 1 + 4} = \frac{-3}{6} = -\frac{1}{2}.$$

Next, we evaluate

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{x^2 - 4}{2x + 4} = -\frac{1}{2}$$

and

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{x^2 - 4}{x^2 - 9x + 14} = \frac{1^2 - 4}{1^2 - 9 \cdot 1 + 14} = -\frac{3}{6} = -\frac{1}{2}.$$

This time, the one-sided limits are equal, so

$$\lim_{x \to 1} f(x) = -1 = f(1).$$

Hence all three parts of the definition of continuity at a point are satisfied, and so $f(x)$ is continuous at $x = 1$.

(c) Since direct substitution of $x = 2$ produces a $\frac{0}{0}$ form, we know that $f(2)$ is undefined, but we must apply the Cancellation Method to determine whether the limit exists:

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - 4}{x^2 - 9x + 14} = \lim_{x \to 2} \frac{(x-2)(x+2)}{(x-2)(x-7)} = \lim_{x \to 2} \frac{x+2}{x-7} = -\frac{4}{5}.$$

Since the limit exists, the discontinuity is removable.

2. We have $f(1) = 2k + 3$, which is defined for all k. By cancellation,

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2 + (k-1)x - k}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+k)}{x-1} = \lim_{x\to 1}(x+k) = 1+k,$$

so the limit exists for all k. In order for $f(x)$ to satisfy the requirement that $\lim_{x\to 1} f(x) = f(1)$, we set

$$1 + k = 2k + 3 \quad \Longrightarrow \quad k = -2,$$

so $f(x)$ is continuous at $x = 1$ only if $k = -2.$

3. First observe that $f(2) = 2k^2 - 5$, which is defined for all k. Since $f(x)$ is a piecewise function whose definition changes at $x = 2$, we investigate the one-sided limits:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{1}{x-4} = -\frac{1}{2} \quad \text{and} \quad \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (k^2 x - 5) = 2k^2 - 5.$$

For the one-sided limits to be equal, we set $2k^2 - 5 = -\frac{1}{2}$ and hence $k = \pm\frac{3}{2}$. Note that for either value of k,

$$f(2) = \lim_{x\to 2} f(x) = -\frac{1}{2},$$

so $f(x)$ is continuous at $x = 2$ for $k = \frac{3}{2}$ and $k = -\frac{3}{2}.$

4. First note that $f(0) = k + \frac{5}{6}$, which is defined for any k. Next,

$$\begin{aligned}\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sqrt{kx^2+1} - 1}{3x^2} &= \lim_{x\to 0} \frac{\sqrt{kx^2+1} - 1}{3x^2} \cdot \frac{\sqrt{kx^2+1} + 1}{\sqrt{kx^2+1} + 1} \\ &= \lim_{x\to 0} \frac{kx^2}{3x^2\left(\sqrt{kx^2+1} + 1\right)} = \lim_{x\to 0} \frac{k}{3\left(\sqrt{kx^2+1} + 1\right)} = \frac{k}{6},\end{aligned}$$

so the limit exists for any k. Finally, we need to determine when $f(0) = \lim_{x\to 0} f(x)$. We set

$$k + \frac{5}{6} = \frac{k}{6} \quad \Longrightarrow \quad \frac{5k}{6} = -\frac{5}{6} \quad \Longrightarrow \quad k = -1,$$

so the only value of k which makes $f(x)$ continuous at $x = 0$ is $k = -1.$