# MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Section 1.5 **Math 1000 Worksheet** Fall 2023

## SOLUTIONS

1. We must first expand the numerator and denominator:

$$f(x) = \frac{x(x+1)(1-x)}{(2x+3)^2} = \frac{-x^3+x}{4x^2+12x+9}.$$

The highest power of $x$ in the denominator is $x^2$, so

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{-x^3+x}{4x^2+12x+9} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{-x+\frac{1}{x}}{4+\frac{12}{x}+\frac{9}{x^2}}.$$

As $x \to \infty$, the numerator tends towards $-\infty$, while the denominator tends towards 4. Hence

$$\lim_{x\to\infty} f(x) = -\infty.$$

As $x \to -\infty$, the numerator tends towards $\infty$, while the denominator again tends towards 4. Thus

$$\lim_{x\to-\infty} f(x) = \infty.$$

2. (a) Since $f(x)$ is a rational function, we only need to take one of the limits at infinity. The highest power of $x$ in the denominator is $x^4$, so

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{6x^3-6x^4}{2x^4-x^2+1} \cdot \frac{\frac{1}{x^4}}{\frac{1}{x^4}} = \lim_{x\to\infty} \frac{\frac{6}{x}-6}{2-\frac{1}{x^2}+\frac{1}{x^4}} = \frac{0-6}{2-0+0} = -3.$$

Hence the only horizontal asymptote of $f(x)$ is the line $y = -3$.

(b) Dividing by $x^3$ in the numerator and denominator gives

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{6x^2-2x+5}{7x^3+x^{\frac{3}{2}}} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to\infty} \frac{\frac{6}{x}-\frac{2}{x^2}+\frac{5}{x^3}}{7+\frac{1}{x^{\frac{3}{2}}}} = \frac{0-0+0}{7+0} = 0.$$

The only horizontal asymptote of $g(x)$ is the line $y = 0$ (the $x$-axis).

(c) First we expand the numerator and denominator:

$$h(x) = \frac{(x+1)^3}{(4x^2+1)(2x-3)} = \frac{x^3+3x^2+3x+1}{8x^3-12x^2+2x-3}.$$

Now we can see that the highest power of $x$ in the denominator is $x^3$, so dividing the numerator and denominator by $x^3$ gives

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} \frac{1+\frac{3}{x}+\frac{3}{x^2}+\frac{1}{x^3}}{8-\frac{12}{x}+\frac{2}{x^2}-\frac{3}{x^3}} = \frac{1+0+0+0}{8-0+0-0} = \frac{1}{8}.$$

The only horizontal asymptote of $h(x)$ is the line $y = \frac{1}{8}$.

3. (a) Observe that the highest power of $x$ in the denominator is $x$, so for both limits we'll divide the numerator and denominator by $x$.

First we take the limit as $x \to \infty$, recalling that for $x > 0$, $x = \sqrt{x^2}$. Hence

$$\lim_{x\to\infty} \frac{x + \sqrt{4x^2+2}}{x-7} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{1 + \frac{\sqrt{4x^2+2}}{\sqrt{x^2}}}{1 - \frac{7}{x}} = \lim_{x\to\infty} \frac{1 + \sqrt{4 + \frac{2}{x^2}}}{1 - \frac{7}{x}}$$

$$= \frac{1 + \sqrt{4+0}}{1-0} = 3.$$

Next we take the limit as $x \to -\infty$, recalling that for $x < 0$, $x = -\sqrt{x^2}$. So then

$$\lim_{x\to-\infty} \frac{x + \sqrt{4x^2+2}}{x-7} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to-\infty} \frac{1 + \frac{\sqrt{4x^2+2}}{-\sqrt{x^2}}}{1 - \frac{7}{x}} = \lim_{x\to-\infty} \frac{1 - \sqrt{4 + \frac{2}{x^2}}}{1 - \frac{7}{x}}$$

$$= \frac{1 - \sqrt{4+0}}{1-0} = -1.$$

Thus the horizontal asymptotes of $f(x)$ are $y = 3$ and $y = -1.$

(b) The largest power of $x$ outside the square root is $x$. The largest power of $x$ inside the square root is $x^2$, but we treat it as if its exponent were reduced by half, making it effectively $x$ as well. Thus we proceed as if the largest power of $x$ in the denominator is $x$, and this is what we will divide the numerator and the denominator by.

First let's take the limit as $x \to \infty$. Recalling that for $x > 0$, $x = \sqrt{x^2}$, we have

$$\lim_{x\to\infty} \frac{2x+1}{5x - \sqrt{9x^2-4}} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{2 + \frac{1}{x}}{5 - \frac{\sqrt{9x^2-4}}{x}} = \lim_{x\to\infty} \frac{2 + \frac{1}{x}}{5 - \frac{\sqrt{9x^2-4}}{\sqrt{x^2}}}$$

$$= \lim_{x\to\infty} \frac{2 + \frac{1}{x}}{5 - \sqrt{9 - \frac{4}{x^2}}} = \frac{2+0}{5 - \sqrt{9-0}} = 1.$$

For the limit as $x \to -\infty$, we follow the same procedure but note that since $x < 0$, $x = -\sqrt{x^2}$. Now we have

$$\lim_{x\to-\infty} \frac{2x+1}{5x - \sqrt{9x^2-4}} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to-\infty} \frac{2 + \frac{1}{x}}{5 - \frac{\sqrt{9x^2-4}}{x}} = \lim_{x\to-\infty} \frac{2 + \frac{1}{x}}{5 - \frac{\sqrt{9x^2-4}}{-\sqrt{x^2}}}$$

$$= \lim_{x\to-\infty} \frac{2 + \frac{1}{x}}{5 + \sqrt{9 - \frac{4}{x^2}}} = \frac{2+0}{5 + \sqrt{9-0}} = \frac{1}{4}.$$

Hence the two horizontal asymptotes are $y = 1$ and $y = \frac{1}{4}.$

(c) Again, the largest power of $x$ outside the square root is $x$ and the largest power inside the square root is effectively $x$ as well.

First let's take the limit as $x \to \infty$. Recalling that for $x > 0$, $x = \sqrt{x^2}$, we have

$$\begin{aligned}\lim_{x\to\infty} \frac{2x+1}{5x-\sqrt{25x^2-4}} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} &= \lim_{x\to\infty} \frac{2+\frac{1}{x}}{5-\frac{\sqrt{25x^2-4}}{x}} = \lim_{x\to\infty} \frac{2+\frac{1}{x}}{5-\frac{\sqrt{25x^2-4}}{\sqrt{x^2}}} \\ &= \lim_{x\to\infty} \frac{2+\frac{1}{x}}{5-\sqrt{25-\frac{4}{x^2}}} = \frac{2+0}{5-\sqrt{25-0}} = \frac{2}{0}.\end{aligned}$$

Since this is a $\frac{K}{0}$ form, this limit does not exist.
For the limit as $x \to -\infty$, we follow the same procedure but note that since $x < 0$, $x = -\sqrt{x^2}$. Now we have

$$\begin{aligned}\lim_{x\to-\infty} \frac{2x+1}{5x-\sqrt{25x^2-4}} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} &= \lim_{x\to-\infty} \frac{2+\frac{1}{x}}{5-\frac{\sqrt{25x^2-4}}{x}} = \lim_{x\to-\infty} \frac{2+\frac{1}{x}}{5-\frac{\sqrt{25x^2-4}}{-\sqrt{x^2}}} \\ &= \lim_{x\to-\infty} \frac{2+\frac{1}{x}}{5+\sqrt{25-\frac{4}{x^2}}} = \frac{2+0}{5+\sqrt{25-0}} = \frac{1}{5}.\end{aligned}$$

Hence the only horizontal asymptotes is $y = \frac{1}{5}$.