MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 1.4 **Math 1000 Worksheet** FALL 2023

SOLUTIONS

1. (a) Direct substitution produces a $\frac{0}{0}$ indeterminate form, so we use the cancellation method:

$$\lim_{x\to 4} \frac{2x^2-7x-4}{3x^2-14x+8} = \lim_{x\to 4} \frac{(2x+1)(x-4)}{(3x-2)(x-4)} = \lim_{x\to 4} \frac{2x+1}{3x-2} = \frac{9}{10},$$

exactly as we deduced using the numerical approach in Question 2(a) of the Worksheet for Section 1.2.

(b) Direct substitution produces a $\frac{0}{0}$ indeterminate form. Since this is a rational function, we use the cancellation method:

$$\lim_{x\to -1} \frac{3x^2-9x-12}{x^3+7x^2+15x+9} = \lim_{x\to -1} \frac{3(x+1)(x-4)}{(x+3)^2(x+1)} = \lim_{x\to -1} \frac{3(x-4)}{(x+3)^2} = \frac{-15}{4}.$$

This corroborates our guess in Question 2(c) of the Worksheet for Section 1.2.

(c) Direct substitution produces a $\frac{0}{0}$ indeterminate form, so we use the cancellation method:

$$\lim_{t\to 2} \frac{t^2-t-6}{t^3-6t^2+12t-8} = \lim_{t\to 2} \frac{(t+3)(t-2)}{(t-2)^3} = \lim_{t\to 2} \frac{t+3}{(t-2)^2}.$$

Now direct substitution produces a $\frac{K}{0}$ form, so the limit does not exist. As $t \to 2$ from either the left or the right, $(t+3)$ tends towards 5 (a positive number) while $(t-2)^2$ becomes a small positive number (because the squares of non-zero real numbers are always positive). Hence

$$\lim_{t\to 2} \frac{t^2-t-6}{t^3-6t^2+12t-8} = \infty.$$

(d) In this case, direct substitution results in a $\frac{K}{0}$ form, so we know that the limit does not exist. As $x \to \frac{1}{2}$ from either side, $3x$ approaches $\frac{3}{2}$ (a positive number). From the left as $x \to \frac{1}{2}$, $(2x-1)$ tends towards a small negative number, so

$$\lim_{x\to \frac{1}{2}^-} \frac{3x}{2x-1} = -\infty.$$

From the right as $x \to \frac{1}{2}$, $(2x-1)$ tends towards a small positive number, so

$$\lim_{x\to \frac{1}{2}^+} \frac{3x}{2x-1} = \infty.$$

Because the one-sided limits do not agree, we cannot assign ∞ or $-\infty$ to the limit.

(e) Direct substitution produces a $\frac{0}{0}$ indeterminate form. This is a quasirational function, so we use the rationalisation method:

$$\begin{aligned}\lim_{x\to-4}\frac{\sqrt{x+8}-2}{x+4}\cdot\frac{\sqrt{x+8}+2}{\sqrt{x+8}+2} &= \lim_{x\to-4}\frac{(x+8)-4}{(x+4)(\sqrt{x+8}+2)}\\ &= \lim_{x\to-4}\frac{x+4}{(x+4)(\sqrt{x+8}+2)}\\ &= \lim_{x\to-4}\frac{1}{\sqrt{x+8}+2} = \frac{1}{4}.\end{aligned}$$

(f) Direct substitution produces a $\frac{0}{0}$ indeterminate form, so we use the rationalisation method:

$$\begin{aligned}\lim_{h\to0}\frac{h^2-h}{\sqrt{h+3}-\sqrt{3}}\cdot\frac{\sqrt{h+3}+\sqrt{3}}{\sqrt{h+3}+\sqrt{3}} &= \lim_{h\to0}\frac{h(h-1)(\sqrt{h+3}+\sqrt{3})}{(h+3)-3}\\ &= \lim_{h\to0}\frac{h(h-1)(\sqrt{h+3}+\sqrt{3})}{h}\\ &= \lim_{h\to0}(h-1)(\sqrt{h+3}+\sqrt{3})\\ &= -2\sqrt{3}.\end{aligned}$$

(g) In this case, we simply need to use direct substitution:

$$\lim_{x\to3}\frac{x-5}{\sqrt{2x+3}+1} = \frac{-2}{\sqrt{9}+1} = -\frac{1}{2}.$$

(h) Direct substitution produces a $\frac{0}{0}$ indeterminate form. We can rid ourselves of the negative exponent in the numerator by multiplying both the numerator and the denominator by $(x+1)$:

$$\frac{12(x+1)^{-1}-2}{x^2-6x+5} = \frac{12-2(x+1)}{(x+1)(x^2-6x+5)} = \frac{-2(x-5)}{(x+1)(x-1)(x-5)}.$$

Now the limit can be rewritten as

$$\begin{aligned}\lim_{x\to5}\frac{12(x+1)^{-1}-2}{x^2-6x+5} &= \lim_{x\to5}\frac{-2(x-5)}{(x+1)(x-1)(x-5)}\\ &= \lim_{x\to5}\frac{-2}{(x+1)(x-1)} = \frac{-2}{24} = -\frac{1}{12}.\end{aligned}$$

(i) Direct substitution yields a $\frac{0}{0}$ indeterminate form. This function can be rewritten in the manner of a normal rational function, which means that we can then use the cancellation method:

$$\begin{aligned}\lim_{h\to0}\frac{\frac{1}{h^2+9}-\frac{1}{9}}{h} = \lim_{h\to0}\frac{\frac{9-(h^2+9)}{9(h^2+9)}}{h} &= \lim_{h\to0}\frac{1}{h}\cdot\frac{-h^2}{9(h^2+9)}\\ &= \lim_{h\to0}\frac{-h}{9(h^2+9)} = \frac{0}{81} = 0.\end{aligned}$$

(j) Direct substitution produces a $\frac{0}{0}$ indeterminate form. The presence of sine functions suggests that we should use the special trigonometric limit. First let's deal with the sine function in the numerator. We need a factor of $8x$ in the denominator, so we write

$$\begin{aligned}\lim_{x\to 0}\frac{\sin(8x)}{\sin(2x)} &= \lim_{x\to 0}\frac{\sin(8x)}{8x}\cdot\frac{8x}{\sin(2x)} = \lim_{x\to 0}\frac{\sin(8x)}{8x}\cdot\lim_{x\to 0}\frac{8x}{\sin(2x)}\\ &= 1\cdot\lim_{x\to 0}\frac{8x}{\sin(2x)} = \lim_{x\to 0}\frac{8x}{\sin(2x)}.\end{aligned}$$

To deal with the remaining sine function, observe that we can factor 4 out of the numerator to obtain a factor of $2x$:

$$\lim_{x\to 0}\frac{\sin(8x)}{\sin(2x)} = 4\lim_{x\to 0}\frac{2x}{\sin(2x)} = 4\lim_{x\to 0}\frac{1}{\left(\frac{\sin(2x)}{2x}\right)} = 4\cdot\frac{1}{1} = 4.$$

Alternatively, we could use the double-angle formula for sine to write

$$\sin(8x) = 2\sin(4x)\cos(4x) = 4\sin(2x)\cos(2x)\cos(4x),$$

so

$$\begin{aligned}\lim_{x\to 0}\frac{\sin(8x)}{\sin(2x)} &= \lim_{x\to 0}\frac{4\sin(2x)\cos(2x)\cos(4x)}{\sin(2x)}\\ &= 4\lim_{x\to 0}\cos(2x)\cos(4x) = 4(1)(1) = 4\end{aligned}$$

by direct substitution.

(k) Direct substitution yields a $\frac{0}{0}$ indeterminate form, so we will use a special trigonometric limit. Observe that

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos^2(x)}{x} &= \lim_{x\to 0}\frac{[1-\cos(x)][1+\cos(x)]}{x}\\ &= \lim_{x\to 0}\frac{1-\cos(x)}{x}\cdot\lim_{x\to 0}[1+\cos(x)]\\ &= 0\cdot 2 = 0.\end{aligned}$$

(ℓ) Direct substitution produces a $\frac{0}{0}$ indeterminate form. We can use the special trigonometric limit:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin(3x^2)}{x\sin(x)} &= \lim_{x\to 0}\frac{3x\sin(3x^2)}{3x^2\sin(x)} = \lim_{x\to 0}3\cdot\frac{x}{\sin(x)}\cdot\frac{\sin(3x^2)}{3x^2}\\ &= 3\lim_{x\to 0}\frac{x}{\sin(x)}\cdot\lim_{x\to 0}\frac{\sin(3x^2)}{3x^2}.\end{aligned}$$

Observe that as $x\to 0$, $3x^2\to 0$ as well, so

$$\lim_{x\to 0}\frac{\sin(3x^2)}{x\sin(x)} = 3(1)(1) = 3.$$

(m) By direct substitution, we obtain

$$\lim_{x\to\pi} \frac{\tan\left(\frac{x}{4}\right)}{1-\cos(x)} = \frac{\tan\left(\frac{\pi}{4}\right)}{1-\cos(\pi)} = \frac{1}{1-(-1)} = \frac{1}{2}.$$

(n) Direct substitution produces a $\frac{0}{0}$ indeterminate form. Because this problem involves secant functions, we need to rewrite it in terms of other trigonometric functions if we're to use a special trigonometric limit. In particular, recall that $\sec(\theta) = \frac{1}{\cos(\theta)}$ so we have

$$\lim_{\theta\to 0} \frac{1-\sec(\theta)}{\theta\sec(\theta)} = \lim_{\theta\to 0} \frac{1-\frac{1}{\cos(\theta)}}{\frac{\theta}{\cos(\theta)}} = \lim_{\theta\to 0} \frac{\frac{\cos(\theta)-1}{\cos(\theta)}}{\frac{\theta}{\cos(\theta)}}$$

$$= \lim_{\theta\to 0} \frac{\cos(\theta)-1}{\theta} = 0.$$

(o) Observe that $|x-2|$ changes its definition at $x=2$:

$$|x-2| = \begin{cases} -(x-2) & \text{if } x < 2 \\ x-2 & \text{if } x \geq 2. \end{cases}$$

Thus we need to examine the one-sided limits. From the left,

$$\lim_{x\to 2^-} \frac{|x-2|-2}{x} = \lim_{x\to 2^-} \frac{-(x-2)-2}{x} = \lim_{x\to 2^-} \frac{-x}{x} = \lim_{x\to 2^-} (-1) = -1.$$

From the right,

$$\lim_{x\to 2^+} \frac{|x-2|-2}{x} = \lim_{x\to 2^+} \frac{(x-2)-2}{x} = \lim_{x\to 2^+} \frac{x-4}{x} = \frac{-2}{2} = -1.$$

Since the one-sided limits agree, we can conclude that

$$\lim_{x\to 2} \frac{|x-2|-2}{x} = -1.$$

(p) Although $x\to -2$, $|x-2|$ does not change its definition at $x=-2$, so we can just substitute directly:

$$\lim_{x\to -2} \frac{|x-2|-2}{x} = \frac{|-4|-2}{-2} = -1.$$

(q) We must check the one-sided limits, since $|x|$ changes its definition at $x=0$. For $x<0$, $|x| = -x$ so we can write

$$\lim_{x\to 0^-} \frac{x^2-4x}{7x-|x|} = \lim_{x\to 0^-} \frac{x^2-4x}{7x-(-x)} = \lim_{x\to 0^-} \frac{x^2-4x}{8x} = \lim_{x\to 0^-} \frac{x-4}{8} = -\frac{1}{2}.$$

For $x>0$, $|x| = x$ so we have

$$\lim_{x\to 0^+} \frac{x^2-4x}{7x-|x|} = \lim_{x\to 0^+} \frac{x^2-4x}{7x-x} = \lim_{x\to 0^+} \frac{x^2-4x}{6x} = \lim_{x\to 0^+} \frac{x-4}{6} = -\frac{2}{3}.$$

Since the one-sided limits are not equal, we can conclude that the given limit does not exist.

2. (a) Since $f(x)$ changes its definition at $x = 1$, we must check the one-sided limits. From the left,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^2 + 3x + 5) = 9.$$

From the right,

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (7x - 2) = 5.$$

Since these are not equal, $\lim_{x\to 1} f(x)$ does not exist.

(b) Again, $g(x)$ changes its definition at $x = 1$, so we must check the one-sided limits. From the left,

$$\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-} (x^2 + 3x + 5) = 9.$$

From the right,

$$\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} (7x + 2) = 9.$$

Since the one-sided limits agree, we can conclude that $\lim_{x\to 1} g(x) = 9$ as well.

(c) This time, $h(x)$ does not change its definition at $x = 1$, so we can simply write

$$\lim_{x\to 1} h(x) = \lim_{x\to 1}(7x - 2) = 5.$$

3. (a) We set the denominator equal to zero, so that

$$x^3 + 3x^2 - 9x + 5 = (x + 5)(x - 1)^2 = 0.$$

Hence the only possible vertical asymptotes are $x = -5$ and $x = 1$.
When $x = -5$, the numerator is $-54 \neq 0$, so we have a $\frac{K}{0}$ form. Hence $x = -5$ is a vertical asymptote. From the left as $x \to -5$, the denominator is a small negative number, so given that the numerator is also negative,

$$\lim_{x\to -5^-} \frac{5x - 4 - x^2}{x^3 + 3x^2 - 9x + 5} = \infty.$$

From the right as $x \to -5$, the denominator is a small positive number, so

$$\lim_{x\to -5^+} \frac{5x - 4 - x^2}{x^3 + 3x^2 - 9x + 5} = -\infty.$$

When $x = 1$, however, the numerator is zero, so we have to take the limit using the cancellation method:

$$\lim_{x\to 1} \frac{5x - 4 - x^2}{x^3 + 3x^2 - 9x + 5} = \lim_{x\to 1} \frac{-(x - 4)(x - 1)}{(x + 5)(x - 1)^2} = \lim_{x\to 1} \frac{4 - x}{(x + 5)(x - 1)}.$$

Now direct substitution produces a $\frac{K}{0}$ form (with $K = 3$) so $x = 1$ is a vertical asymptote after all. From the left as $x \to 1$, the denominator is a small negative number, so

$$\lim_{x\to 1^-} \frac{5x - 4 - x^2}{x^3 + 3x^2 - 9x + 5} = -\infty.$$

From the right as $x \to 1$, the denominator is a small positive number, so

$$\lim_{x\to 1^+} \frac{5x-4-x^2}{x^3+3x^2-9x+5} = \infty.$$

(b) We set the denominator equal to zero, so that

$$5x - x^2 - 4 = -(x-4)(x-1) = 0.$$

Hence the only possible vertical asymptotes are $x = 4$ and $x = 1$.
When $x = 4$, the numerator is $81 \neq 0$, so we have a $\frac{K}{0}$ form. Hence $x = 4$ is a vertical asymptote. From the left as $x \to 4$, the denominator is a small positive number, so

$$\lim_{x\to 4^-} \frac{x^3+3x^2-9x+5}{5x-4-x^2} = \infty.$$

From the right as $x \to 4$, the denominator is a small negative number, so

$$\lim_{x\to 4^+} \frac{x^3+3x^2-9x+5}{5x-4-x^2} = -\infty.$$

When $x = 1$, however, the numerator is zero, so we take the limit using the cancellation method:

$$\begin{aligned}\lim_{x\to 1} \frac{x^3+3x^2-9x+5}{5x-4-x^2} &= \lim_{x\to 1} \frac{(x+5)(x-1)^2}{-(x-4)(x-1)} \\ &= \lim_{x\to 1} \frac{(x+5)(x-1)}{4-x} = 0.\end{aligned}$$

Because $\lim\limits_{x\to 1} f(x)$ exists, $x = 1$ is not a vertical asymptote.

4. Using the inequality, we can write

$$-\cot(x) \leq \cot(x)\sin\left(\frac{1}{x}\right) \leq \cot(x)$$

if $\cot(x) > 0$ or

$$-\cot(x) \geq \cot(x)\sin\left(\frac{1}{x}\right) \geq \cot(x)$$

if $\cot(x) < 0$. Furthermore,

$$\lim_{x\to\frac{\pi}{2}} \cot(x) = \cot\left(\frac{\pi}{2}\right) = 0 \quad \text{and} \quad \lim_{x\to\frac{\pi}{2}} -\cot(x) = -\cot\left(\frac{\pi}{2}\right) = 0.$$

Thus, by the Squeeze Theorem, we conclude that $\lim\limits_{x\to\frac{\pi}{2}} \cot(x)\sin\left(\frac{1}{x}\right) = 0$ as well.

(Note that if you're not comfortable evaluating a cotangent directly, you can always use the identity $\cot(x) = \dfrac{\cos(x)}{\sin(x)}$. Here, for instance, $\cot\left(\dfrac{\pi}{2}\right) = \dfrac{\cos\left(\frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{2}\right)} = \dfrac{0}{1} = 0$.)

5. We know that

$$-1 \leq \cos\left(\frac{\pi}{2x}\right) \leq 1.$$

If we multiply all parts of the inequality by some $x > 0$, we get

$$-x \leq x\cos\left(\frac{\pi}{2x}\right) \leq x.$$

On the other hand, if $x < 0$, the same multiplication flips the direction of the inequalities, giving

$$x \leq x\cos\left(\frac{\pi}{2x}\right) \leq -x.$$

We can combine these two cases if we recall that $|x| = x$ for $x > 0$ and $|x| = -x$ for $x < 0$. Thus we have

$$-|x| \leq x\cos\left(\frac{\pi}{2x}\right) \leq |x|.$$

We know that $\lim_{x\to 0}|x| = 0$ and so

$$\lim_{x\to 0} -|x| = -\lim_{x\to 0}|x| = 0$$

as well. By the Squeeze Theorem, then, we also have

$$\lim_{x\to 0} x\cos\left(\frac{\pi}{2x}\right) = 0.$$