Section 4.3: Curve Sketching

We can use calculus to develop many pieces of information about the graph of a function, and therefore to help sketch an accurate graph of the resulting curve.

Thus given a function $f(x)$ we consider each of the following to help sketch the curve:

(1) Identify the domain of $f(x)$, including any vertical asymptotes and other points of discontinuity.

(2) Find any horizontal asymptotes.

(3) Determine any x-intercepts (the points where $f(x) = 0$) and the y-intercept, if any (the point given by $f(0)$).

(4) Use $f'(x)$ to find the intervals on which $f(x)$ is increasing or decreasing, and to classify any critical points as relative maxima or minima.

(5) Use $f''(x)$ to find the intervals on which $f(x)$ is concave upward or concave downward, and to classify any hypercritical points as points of inflection (if appropriate).

(6) Finally, use all of this information to sketch the graph of $y = f(x)$.

eg Sketch the graph of $f(x) = \dfrac{x^4 - 2x^3}{9x^2 - 36}$.

① We set $9x^2 - 36 = 0$

$$9(x^2 - 4) = 0$$

$$9(x-2)(x+2) = 0 \rightarrow x = 2,\ x = -2$$

$f(2) = \frac{0}{0}$ so we evaluate

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^3(x-2)}{9(x-2)(x+2)}$$

$$= \lim_{x \to 2} \frac{x^3}{9(x+2)} = \frac{8}{36} = \frac{2}{9}$$

There is a removable discontinuity at $(2, 2/9)$.

$f(-2) = \frac{32}{0}$ so $x = -2$ is a vertical asymptote

② $$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^4-2x^3}{9x^2-36}\cdot\frac{1/x^2}{1/x^2}$$

$$= \lim_{x\to\infty} \frac{x^2-2x}{9-36/x^2}$$

$$= \infty$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^4-2x^3}{9x^2-36}\cdot\frac{1/x^2}{1/x^2}$$

$$= \lim_{x\to\infty} \frac{x^2-2x}{9-36/x^2}$$

$$= \infty$$

There are <u>no</u> horizontal asymptotes.

③ We set

$$f(x)=0$$

$$x^4-2x^3=0$$

$$x^3(x-2)=0 \rightarrow x=0,\ x=2$$

NOT IN DOMAIN

The only x-intercept is $(0,0)$ which is also the y-intercept.

④ $$f'(x) = \frac{2x^2(x+3)}{9(x+2)^2}$$

Note that $f'(x)$ is undefined only at $x=-2$ (the vertical asymptote) so we set

$$f'(x)=0$$

$$2x^2(x+3)=0 \rightarrow x=0,\ x=-3$$

(CRITICAL POINTS)

$f'(x)$	$-$		$+$	VA	$+$		$+$	RD	$+$
$f(x)$	$\searrow$	-3	$\nearrow$	-2	$\nearrow$	0	$\nearrow$	2	$\nearrow$

$f(x)$ is increasing for $-3 < x < -2$, $-2 < x < 0$, $0 < x < 2$, $x > 2$

decreasing for $x < -3$

$x = -3$ is a relative minimum; this is the point $(-3, 3)$

$x = 0$ is a saddle point; this is the point $(0, 0)$

(5) $$f''(x) = \frac{2x(x^2+6x+12)}{9(x+2)^3}$$

$f''(x)$ is undefined only at the vertical asymptote $x = -2$, so we set

$$f''(x) = 0$$

$$2x(x^2+6x+12) = 0$$

$x = 0$ $\qquad$ $x^2 + 6x + 12 = 0$

$\nearrow$ $\qquad$ $x = \dfrac{-6 \pm \sqrt{36 - 48}}{2}$ which is undefined

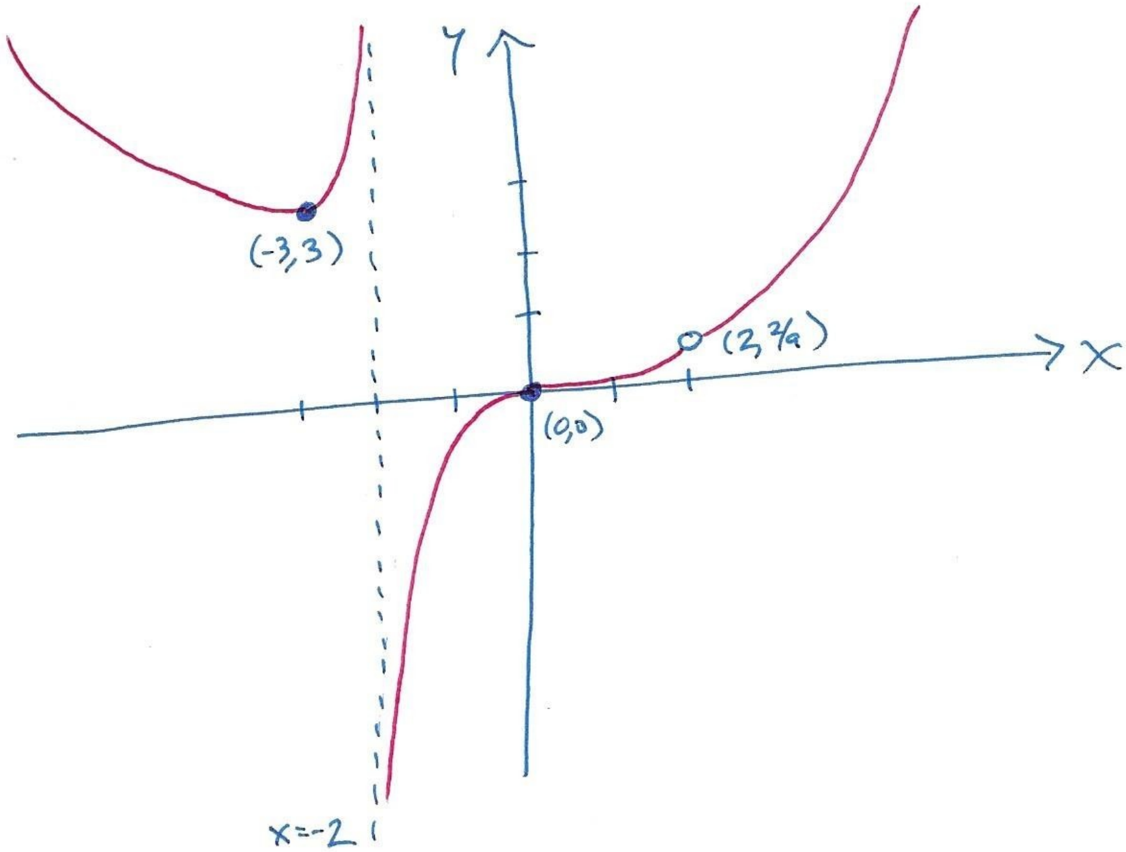
ONLY HYPERCRITICAL POINT

$f''(x)$	$+$	VA	$-$		$+$	RD	$+$
$f(x)$	$\cup$	-2	$\cap$	0	$\cup$	2	$\cup$

$f(x)$ is concave upward for $x < -2$, $0 < x < 2$, $x > 2$

concave downward for $-2 < x < 0$

$x = 0$ is a point of inflection; this is the point $(0, 0)$



Y
X
(-3,3)
(0,0)
(2, 2/9)
x=-2