# Section 4.1: Related Rates

In most applications, we consider processes which can be described by a function which changes with time, $y=f(t)$. Then its derivative $\frac{dy}{dt}$ represents the rate of change of the process with time, so if $\frac{dy}{dt} > 0$ then $y$ is growing in time, while if $\frac{dy}{dt} < 0$ then $y$ is shrinking in time. If $\frac{dy}{dt} = 0$ then $y$ is constant, and not changing in time.

In many situations, we will encounter two or more processes whose rates of change are related. This is called a <u>related rates problem</u>.

eg Assume that $x$ and $y$ are real numbers related by the equation $xy = 10$. Given that $\frac{dx}{dt} = -2$, find $\frac{dy}{dt}$ when $x=5$.

We need to find the relationship between the derivatives of the quantity in the problem, here $x$ and $y$. To do so, we will differentiate the equation relating the variables

$$\frac{d}{dt}[xy] = \frac{d}{dt}[10]$$

$$\frac{d}{dt}[x]\cdot y + x\cdot\frac{d}{dt}[y] = 0$$

$$\frac{dx}{dt}\cdot y + x\cdot\frac{dy}{dt} = 0 \qquad (*)$$

When $x=5$, since $xy=10$, then

$$5\cdot y = 10 \rightarrow y=2.$$

Hence, by Equation $(*)$,

$$-2\cdot 2 + 5\cdot\frac{dy}{dt} = 0$$

$$5\cdot\frac{dy}{dt} = 4$$

$$\boxed{\frac{dy}{dt} = \frac{4}{5}}$$

Strategy for related rates problems

① Identify the variables of the problem, any given quantities, and the rate of change to be found.

② Establish an equation which relates the variables of the problem.

③ Use implicit differentiation to obtain Equation $(*)$, which relates the rates of change of the problem.

(4) Substitute all known values into Equation (*) and solve for the desired rate of change.

In many problems, the equation which relates the variables of the problem will be geometric nature, including formulas for area, volume, perimeter, surface area, as well as the Pythagorean theorem and the properties of similar triangles.

eg A spherical soap bubble is absorbing 10 $cm^3$ of air every second. How quickly is the radius of the bubble changing at the moment when it measures 1 cm?

Let $V$ be the volume of the bubble

$r$ be the radius of the bubble

We are given that $\frac{dV}{dt} = 10$.

We want to find $\frac{dr}{dt}$ when $r = 1$.

We know that

$$V = \frac{4}{3}\pi r^3$$

We differentiate implicitly:

$$\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{4}{3}\pi r^3\right]$$

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt} \qquad (*)$$

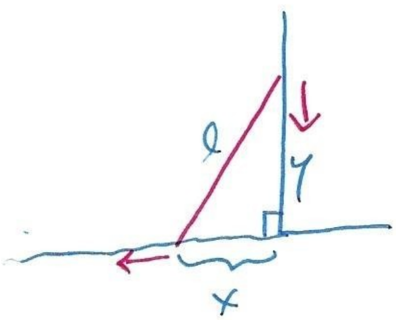
Now we substitute into Equation $(*)$:

$$10 = 4\pi \cdot 1^2 \cdot \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{10}{4\pi} = \frac{5}{2\pi}$$

The radius of the bubble changes at $\frac{5}{2\pi}$ cm/sec.

eg A 25-foot-tall ladder is leaning against the exterior wall of a house. Its base is being pulled away from the wall at a rate of 2 ft/sec. How fast is the top of the ladder sliding down the wall when its base is 7 ft from the wall?




Let $\ell$ be the length of the ladder, let $x$ be the distance from its base to the wall, and let $y$ be the the distance from its top to the ground.

We are given that $\ell = 25$ (constant), and that $\frac{dx}{dt} = 2$. We want to find $\frac{dy}{dt}$ when $x = 7$.

By the Pythagorean theorem,

$$x^2 + y^2 = \ell^2$$

$$\frac{d}{dt}\left[x^2+y^2\right] = \frac{d}{dt}\left[\ell^2\right]$$

$$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0$$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0 \qquad (*)$$

When $x = 7$ and $l = 25$,

$$x^2 + y^2 = l^2$$

$$7^2 + y^2 = 25^2 \rightarrow y = 24$$

Substituting into Equation $(*)$, we have

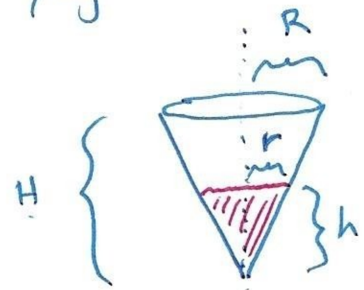
$$7 \cdot 2 + 24 \cdot \frac{dy}{dt} = 0$$

$$24 \cdot \frac{dy}{dt} = -14$$

$$\frac{dy}{dt} = \frac{-14}{24} = -\frac{7}{12}$$

The top of the ladder slides down the wall at a rate of $\frac{7}{12}$ ft / sec.

eg An inverted cone is being filled with yoghurt. The cone is 12 cm high and has a radius of 4 cm at the top. If the volume of yoghurt in the cone increases at 0.4 cm$^3$/sec, determine how quickly the yoghurt is rising when it is 2 cm deep.




Let H be the height of the cone, and R be its radius.
Let V be the volume of the yoghurt and h be its height, and r be its radius.

We are given that $H = 12$ (constant), $R = 4$ (constant), and $\frac{dV}{dt} = 0.4$.

We want to find $\frac{dh}{dt}$ when $h = 2$.

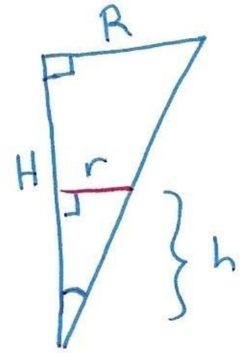
We know that

$$V = \frac{\pi}{3} r^2 h$$

$$\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{\pi}{3} r^2 h\right]$$

$$\frac{dV}{dt} = \frac{\pi}{3} \cdot 2r \frac{dr}{dt} \cdot h + \frac{\pi}{3} r^2 \cdot \frac{dh}{dt} \qquad (*)$$

However, we have no information about $\frac{dr}{dt}$ so this is not a useful Equation (*).




The triangle representing the cone is similar to the triangle representing the yoghurt. Thus

$$\frac{r}{R} = \frac{h}{H}$$

$$r = \frac{Rh}{H}$$

Therefore we can rewrite the equation for the volume as

$$V = \frac{\pi}{3} \cdot \left(\frac{Rh}{H}\right)^2 \cdot h$$

$$= \frac{\pi}{3} \cdot \frac{R^2 h^3}{H^2}$$

$$\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{\pi}{3} \cdot \frac{R^2 h^3}{H^2}\right]$$

$$\frac{dV}{dt} = \frac{\pi}{3} \cdot \frac{R^2}{H^2} \cdot 3h^2 \frac{dh}{dt}$$

$$\frac{dV}{dt} = \pi \cdot \frac{R^2}{H^2} \cdot h^2 \frac{dh}{dt} \qquad (*)$$

Now we substitute into the new Equation $(*)$:

$$0.4 = \pi \cdot \frac{4^2}{12^2} \cdot 2^2 \frac{dh}{dt}$$

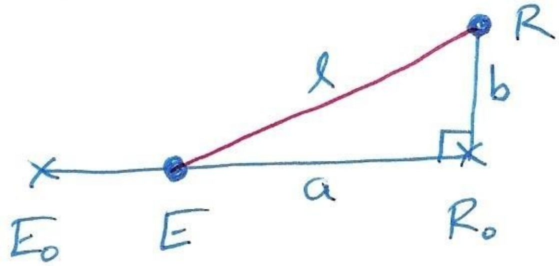
$$\frac{dh}{dt} = \frac{9}{10\pi}$$

The yoghurt is rising at a rate of $\underline{\frac{9}{10\pi} \text{ cm/sec}}$.

eg Marshal Wyatt Earp is in hot pursuit of notorious outlaw Johnny Ringo. At 9:00am, Earp is 200 km west of Ringo's position, and is riding east at a constant speed of 50 km/hr. Ringo starts heading north at 40 km/hr. To one decimal place, how quickly is the distance between them changing at high noon?

AT 9:00am:


AFTER 9:00am:




Let a be the distance between Earp's current position and Ringo's original position.

Let b be the distance between Ringo's original and current positions.

Let $l$ be the distance that currently separates Earp and Ringo.

We are given that $\frac{da}{dt} = -50$ and $\frac{db}{dt} = 40$.

We want to find $\frac{dl}{dt}$ at 12:00.

By the Pythagorean theorem,

$$a^2 + b^2 = \ell^2$$

$$\frac{d}{dt}\left[a^2+b^2\right] = \frac{d}{dt}\left[\ell^2\right]$$

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2\ell\frac{d\ell}{dt}$$

$$a\frac{da}{dt} + b\frac{db}{dt} = \ell\frac{d\ell}{dt} \qquad (*)$$

At 12:00, $a = 200 - 50 \cdot 3 = 200 - 150$

$= 50$

$b = 40 \cdot 3 = 120$

$\ell^2 = a^2 + b^2 = 50^2 + 120^2$

$\ell = 130$

We substitute into Equation $(*)$:

$$50 \cdot (-50) + 120 \cdot 40 = 130 \cdot \frac{d\ell}{dt}$$

$$\frac{d\ell}{dt} = \frac{230}{13} \approx 17.7$$

The distance between Earp and Ringo is increasing at a rate of approximately 17.7 km/hr.