Section 3.1: Derivatives of Exponential and Trigonometric Functions

In general, an exponential function has the form $f(x) = b^x$ where b is any real number. For our purposes, we will consider $b > 1$ only.

Then
$$\frac{d}{dx}[b^x] = \lim_{h \to 0} \frac{b^{x+h} - b^x}{h}$$
$$= \lim_{h \to 0} \frac{b^x \cdot b^h - b^x}{h}$$
$$= b^x \cdot \lim_{h \to 0} \frac{b^h - 1}{h}$$

Using the numerical method, we can try to evaluate $\lim_{h \to 0} \frac{b^h - 1}{h}$ for different $b > 1$:

b	$3/2$	2	$5/2$	3	$7/2$	4
$\lim_{h \to 0} \frac{b^h - 1}{h}$	0.405	0.693	0.916	1.10	1.253	1.386

While we cannot draw any obvious conclusions about $\lim_{h \to 0} \frac{b^h - 1}{h}$ in general, we can see that there is a value of b between $5/2$ and 3 for which
$$\lim_{h \to 0} \frac{b^h - 1}{h} = 1.$$

This special value is called Euler's constant, denoted by

$$e \approx 2.7182818284\ldots$$

The exponential function e^x is called the natural exponential function, also denoted by $\exp(x)$.

We now have that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$ so

$$[e^x]' = e^x \cdot \lim_{h \to 0} \frac{e^h - 1}{h} = e^x \cdot 1 = e^x.$$

eg $y = e^x \sqrt{x}$

$$\frac{dy}{dx} = \frac{d}{dx}[e^x] \cdot \sqrt{x} + e^x \cdot \frac{d}{dx}[\sqrt{x}]$$

$$= e^x \sqrt{x} + e^x \cdot \frac{1}{2} x^{-1/2}$$

$$\boxed{= e^x \sqrt{x} + \frac{e^x}{2\sqrt{x}}}$$

The logarithmic function with base e is called the natural logarithmic function. It is denoted by $\log_e(x) = \ln(x)$. Observe that

$$e^{\ln(x)} = x \quad \text{and} \quad \ln(e^x) = x.$$

<u>Theorem</u>: $[\sin(x)]' = \cos(x)$

<u>Proof</u>: $$[\sin(x)]' = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

Recall that
$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$
so $$\sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h).$$

Now we have
$$[\sin(x)]' = \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$
$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) - \sin(x)}{h} + \lim_{h\to 0} \frac{\cos(x)\sin(h)}{h}$$
$$= \sin(x) \cdot \lim_{h\to 0} \frac{\cos(h) - 1}{h} + \cos(x) \cdot \lim_{h\to 0} \frac{\sin(h)}{h}$$
$$= \sin(x)\cdot 0 + \cos(x)\cdot 1$$ by the Special Trig Limits
$$= \cos(x).$$

eg $f(x) = x^4 \sin(x)$
$$f'(x) = [x^4]' \sin(x) + x^4 [\sin(x)]'$$
$$\boxed{= 4x^3 \sin(x) + x^4 \cos(x)}$$

<u>Theorem</u>: $[\cos(x)]' = -\sin(x)$

__Theorem__: $[\sec(x)]' = \sec(x)\tan(x)$

$[\csc(x)]' = -\csc(x)\cot(x)$

__Proof__: We will prove $[\sec(x)]'$. Recall that

$$\sec(x) = \frac{1}{\cos(x)}.$$

Then
$$[\sec(x)]' = \frac{[1]'\cos(x) - 1\cdot[\cos(x)]'}{[\cos(x)]^2}$$

$$= \frac{0\cdot\cos(x) - [-\sin(x)]}{\cos^2(x)}$$

$$= \frac{\sin(x)}{\cos^2(x)}$$

$$= \frac{1}{\cos(x)}\cdot\frac{\sin(x)}{\cos(x)} = \sec(x)\tan(x).$$

eg $f(x) = \cos(x) + \csc(x)$

$f'(x) = -\sin(x) + [-\csc(x)\cot(x)]$

$\boxed{= -\sin(x) - \csc(x)\cot(x)}$

eg $f(t) = \dfrac{5}{\sec(t)} = 5\cdot\dfrac{1}{\sec(t)} = 5\cos(t)$

$f'(t) = 5[-\sin(t)]$

$\boxed{= -5\sin(t)}$

<u>Theorem</u>: $[\tan(x)]' = \sec^2(x)$

$$[\cot(x)]' = -\csc^2(x)$$

<u>Proof</u>: We will prove $[\cot(x)]'$. We write

$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

so
$$[\cot(x)]' = \frac{[\cos(x)]'\sin(x) - \cos(x)[\sin(x)]'}{[\sin(x)]^2}$$

$$= \frac{[-\sin(x)]\sin(x) - \cos(x)\cdot\cos(x)}{\sin^2(x)}$$

$$= \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)}$$

$$= -\frac{[\sin^2(x) + \cos^2(x)]}{\sin^2(x)}$$

$$= -\frac{1}{\sin^2(x)} = -\csc^2(x).$$

eg $y = xe^x \tan(x)$

$$y' = [x]'e^x\tan(x) + x\cdot[e^x\tan(x)]'$$

$$= 1\cdot e^x\tan(x) + x\cdot\left([e^x]'\tan(x) + e^x\cdot[\tan(x)]'\right)$$

$$= e^x\tan(x) + x\cdot\left(e^x\tan(x) + e^x\sec^2(x)\right)$$

$$\boxed{= e^x\tan(x) + xe^x\tan(x) + xe^x\sec^2(x)}$$

eg Find the equation of the tangent line to $y = \sec(x) - 2\cos(x)$ at $x = \frac{\pi}{3}$.

When $x = \frac{\pi}{3}$, $y = \sec\left(\frac{\pi}{3}\right) - 2\cos\left(\frac{\pi}{3}\right)$

$$= 2 - 2 \cdot \tfrac{1}{2} = 1$$

Next, $y' = [\sec(x)]' - 2[\cos(x)]'$

$$= \sec(x)\tan(x) - 2[-\sin(x)]$$

$$= \sec(x)\tan(x) + 2\sin(x)$$

At $x = \frac{\pi}{3}$, $y' = \sec\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{3}\right) + 2\sin\left(\frac{\pi}{3}\right)$

$$= 2 \cdot \sqrt{3} + 2 \cdot \frac{\sqrt{3}}{2} = 3\sqrt{3}$$

We will use point-slope form to get the equation of the tangent line:

$$y - y_0 = m(x - x_0)$$

$$y - 1 = 3\sqrt{3}\left(x - \frac{\pi}{3}\right)$$

$$\boxed{y = 3\sqrt{3}x - \pi\sqrt{3} + 1}$$

The normal line to a curve at a point is the line that is perpendicular to the tangent line at that point.

Let m_T be the slope of the tangent line, and let m_N be the slope of the normal line. Then

$$m_N = -\frac{1}{m_T}.$$

If $m_T = 0$, meaning the tangent line is horizontal, then the normal line is vertical. The reverse is also true.

eg Find the equation of the normal line to $y = \sec(x) - 2\cos(x)$ at $x = \frac{\pi}{3}$.

Here, $m_T = 3\sqrt{3}$ so

$$m_N = -\frac{1}{3\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$

$$= -\frac{\sqrt{3}}{9}$$

Thus the equation of the normal line is

$$y - 1 = -\frac{\sqrt{3}}{9}\left(x - \frac{\pi}{3}\right)$$

$$\boxed{y = -\frac{\sqrt{3}}{9}x + \frac{\pi\sqrt{3}}{27} + 1}$$