Section 2.3: Derivatives of Algebraic Functions

We will try to develop rules for differentiation which do not require us to use the limit def'n every time we want to find a derivative.

Theorem: The Constant Rule

For any real number k, $\frac{d}{dx}[k] = 0$.

Proof: $$\frac{d}{dx}[k] = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{where } f(x) \equiv k$$

$$= \lim_{h \to 0} \frac{k - k}{h}$$

$$= \lim_{h \to 0} 0 \boxed{= 0}$$

Theorem: $\frac{d}{dx}[x] = 1$

Proof: $\frac{d}{dx}[x] = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$ where $f(x) = x$

$$= \lim_{h\to 0} \frac{(x+h) - x}{h}$$

$$= \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 \boxed{= 1}$$

Theorem: The Power Rule

For any real number r,

$$\frac{d}{dx}[x^r] = r x^{r-1}$$

eg $[x^2]' = 2x^{2-1} \boxed{= 2x}$

$$[\sqrt{x}]' = [x^{1/2}]' = \frac{1}{2} x^{1/2 - 1} = \frac{1}{2} x^{-1/2}$$

$$= \frac{1}{2x^{1/2}} \boxed{= \frac{1}{2\sqrt{x}}}$$

$$\left[\frac{1}{x}\right]' = [x^{-1}]' = (-1)\cdot x^{-1-1} \boxed{\begin{aligned} &= -x^{-2} \\ &= \frac{-1}{x^2} \end{aligned}}$$

$$[x^{99}]' = 99x^{99-1} \boxed{= 99x^{98}}$$

eg Differentiate $f(x) = \left(\frac{1}{\sqrt[3]{x}}\right)^4$.

Rewrite: $f(x) = \left(\frac{1}{x^{1/3}}\right)^4$

$= \left(x^{-1/3}\right)^4$

$= x^{-4/3}$

Differentiate: $\boxed{f'(x) = -\frac{4}{3} x^{-7/3}}$

eg Find the derivative of $f(x) = \pi^3$.

This is a constant function, so we apply the Constant Rule: $\boxed{f'(x) = 0}$

Theorem: The Constant Multiple Rule

For any real number k,

$$[kf(x)]' = kf'(x).$$

Proof: Let $g(x) = kf(x)$ so

$$[kf(x)]' = g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{kf(x+h) - kf(x)}{h}$$

$$= \lim_{h \to 0} \frac{k[f(x+h) - f(x)]}{h}$$

$$= k \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= k f'(x).$$

eg Differentiate $y = -5x^7$.

$$\frac{dy}{dx} = -5 \cdot \frac{d}{dx}\left[x^7\right]$$

$$= -5 \cdot (7x^6)$$

$$\boxed{= -35x^6}$$

eg Differentiate $f(t) = \frac{\sqrt{t}}{8}$.

Rewrite: $f(t) = \frac{1}{8} \cdot \sqrt{t} = \frac{1}{8} \cdot t^{1/2}$

Differentiate: $f'(t) = \frac{1}{8} \cdot \left[t^{1/2}\right]'$

$$= \frac{1}{8} \cdot \frac{1}{2} t^{-1/2}$$

$$\boxed{\begin{aligned} &= \frac{1}{16} t^{-1/2} \\ &= \frac{1}{16\sqrt{t}} \end{aligned}}$$

Section 2.3

Basic Strategy for Differentiation

1. Given a function $f(x)$ to differentiate, rewrite $f(x)$ in a more suitable form (if appropriate).
2. Apply all necessary differentiation rules.
3. Simplify the result (if possible).

Theorem: The Sum Rule

$$\frac{d}{dx}\left[f(x) + g(x)\right] = \frac{d}{dx}\left[f(x)\right] + \frac{d}{dx}\left[g(x)\right]$$

Proof: Let $A(x) = f(x) + g(x)$. Then

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[A(x)]$$

$$= \lim_{h\to 0} \frac{A(x+h)-A(x)}{h}$$

$$= \lim_{h\to 0} \frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)+g(x+h)-g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$= \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

eg $y = \sqrt{x^3} + \frac{1}{\sqrt{x^3}}$

Rewrite: $y = x^{3/2} + x^{-3/2}$

Differentiate: $\frac{dy}{dx} = \frac{d}{dx}\left[x^{3/2}\right] + \frac{d}{dx}\left[x^{-3/2}\right]$

$$= \frac{3}{2}x^{1/2} + \left(-\frac{3}{2}\right)x^{-5/2}$$

Simplify: $\boxed{= \frac{3}{2}\sqrt{x} - \frac{3}{2}x^{-5/2}}$

Theorem: The Difference Rule

$$[f(x)-g(x)]' = f'(x) - g'(x)$$

Proof: We can write

$$f(x) - g(x) = f(x) + (-1)\,g(x)$$

By the Sum Rule,

$$[f(x)-g(x)]' = [f(x)]' + [(-1)g(x)]'$$

$$= f'(x) + (-1)\,g'(x) \quad \text{by the Const. Multiple Rule}$$

$$= f'(x) - g'(x).$$

eg $f(t) = (t-2)\sqrt[3]{t}$

Rewrite: $f(t) = (t-2)\,t^{1/3}$

$$= t^{4/3} - 2t^{1/3}$$

Differentiate: $f'(t) = [t^{4/3}]' - [2t^{1/3}]'$

$$= \frac{4}{3}t^{1/3} - 2\cdot\frac{1}{3}t^{-2/3}$$

Simplify: $\boxed{= \frac{4}{3}t^{1/3} - \frac{2}{3}t^{-2/3}}$

We can apply the Constant Rule, Power Rule, Constant Multiple Rule, Sum Rule and Difference Rule to differentiate <u>any</u> polynomial function.

eg $p(x) = x^5 - \frac{1}{2}x^2 + 4x + 7$

$$p'(x) = 5x^4 - \frac{1}{2}\cdot 2x + 4\cdot 1 + 0$$
$$\boxed{= 5x^4 - x + 4}$$

We now that if an object moves in a straight line with position $s(t)$ and velocity $v(t)$ then

$$v(t) = s'(t) = \frac{d}{dt}[s(t)].$$

eg A ball is thrown upwards with a velocity of 80 ft/sec. Then its height after t seconds is

$$s(t) = 80t - 16t^2,$$

measured in feet. What is its velocity after 4 sec? What maximum height does it achieve?

$$v(t) = s'(t) = [80t - 16t^2]'$$
$$= 80\cdot 1 - 16\cdot 2t$$
$$= 80 - 32t$$

$$v(4) = 80 - 32\cdot 4 = -48$$

The velocity of the ball after 4 sec is <u>48 ft/sec (downwards)</u>.

The ball reaches its maximum height at the instant when $v(t) = 0$. Thus we set

$$80 - 32t = 0$$

$$t = \frac{80}{32} = \frac{5}{2} = 2.5$$

Then $s(2.5) = 80 \cdot 2.5 - 16(2.5)^2 = 100.$

The maximum height is $\underline{100 \text{ ft}}$.