Section 1.4: Techniques for Evaluating Limits

There are several indeterminate forms: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \cdot \infty$, $\infty - \infty$, 0^0, ∞^0 and 1^∞.

eg $\lim_{x \to 2} \frac{x^2-4}{x^2-3x+2}$

Direct substitution produces a $\frac{0}{0}$ form.

Observe that

$$\frac{x^2-4}{x^2-3x+2} = \frac{(x-2)(x+2)}{(x-2)(x-1)}$$

$$= \frac{x+2}{x-1} \quad \text{for } x \neq 2$$

Thus $\lim_{x \to 2} \frac{x^2-4}{x^2-3x+2} = \lim_{x \to 2} \frac{x+2}{x-1}$

$$= \frac{4}{1}$$

$$\boxed{= 4}$$

This is the <u>cancellation method</u> for the limits of rational functions. Given $\lim_{x \to p} f(x)$ where $f(x)$ is a rational function, if direct substitution yields a $\frac{0}{0}$ form then we should be able to factor $(x-p)$ out of the numerator and the denominator. We cancel these common factors, and try direct substitution again.

eg $\lim_{x\to 3} \dfrac{x^3-27}{3x-x^2}$ $\left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x\to 3} \frac{(x-3)(x^2+3x+9)}{-x(x-3)}$$

$$= \lim_{x\to 3} \frac{x^2+3x+9}{-x} = \frac{27}{-3} \boxed{= -9}$$

Given a rational function $f(x) = \frac{P(x)}{Q(x)}$, we can find all the vertical asymptotes to its graph as follows:

① Find all solutions $x = p$ to the equation $Q(x) = 0$.

② For each $x = p$, evaluate $f(p)$.

→ If a $\frac{K}{0}$ form results, $x = p$ must be a vertical asymptote.

→ If a $\frac{0}{0}$ form results, apply the cancellation method to $\lim_{x \to p} f(x)$. If this now results in a $\frac{K}{0}$ form, $x = p$ must be a vertical asymptote.

eg Find all the vertical asymptotes to the graph of

$$f(x) = \frac{x^2 + 2x + 1}{x^2 - x - 2}.$$

We set

$$x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$$x = 2 \quad x = -1$$

(two possible vertical asymptotes)

For $x=2$, $f(2)=\frac{9}{0}$ which is a $\frac{K}{0}$ form and hence $x=2$ __is__ a vertical asymptote.

For $x=-1$, $f(-1)=\frac{0}{0}$ so we compute

$$\lim_{x\to -1}\frac{x^2+2x+1}{x^2-x-2} = \lim_{x\to -1}\frac{(x+1)^2}{(x-2)(x+1)}$$

$$= \lim_{x\to -1}\frac{x+1}{x-2}$$

$$= \frac{0}{-3}\ \boxed{=0}$$

Since the limit exists, $x=-1$ is __not__ a vertical asymptote.

Hence $x=2$ is the __only__ vertical asymptote.

__Def'n__: A __quasirational__ function is a function that is similar to a rational function except that some of the polynomial terms are contained within radicals.

eg $f(x)=\dfrac{x-1}{\sqrt{x^2+3}-2}$ is a quasirational function

Now we consider the case of the limit of a quasirational function where direct substitution yields a $\frac{0}{0}$ form.

$$\text{eg} \quad \lim_{x \to 1} \frac{x-1}{\sqrt{x^2+3} - 2} \qquad \left(\frac{0}{0} \text{ form}\right)$$

We will try multiplying the numerator and the denominator by the <u>conjugate</u> of the denominator.

Recall that the conjugate of an expression $\sqrt{A} - B$ is $\sqrt{A} + B$, and vice versa.

Then we have

$$\frac{x-1}{\sqrt{x^2+3} - 2} \cdot \frac{\sqrt{x^2+3} + 2}{\sqrt{x^2+3} + 2}$$

$$= \frac{(x-1)(\sqrt{x^2+3} + 2)}{(x^2+3) + 2\sqrt{x^2+3} - 2\sqrt{x^2+3} - 4}$$

$$= \frac{(x-1)(\sqrt{x^2+3} + 2)}{x^2 - 1}$$

$$= \frac{(x-1)(\sqrt{x^2+3} + 2)}{(x-1)(x+1)} = \frac{\sqrt{x^2+3} + 2}{x+1}$$

Now we have

$$\lim_{x\to 1} \frac{x-1}{\sqrt{x^2+3}-2} = \lim_{x\to 1} \frac{\sqrt{x^2+3}+2}{x+1}$$

$$= \frac{4}{2}$$

$$\boxed{= 2}$$

This is the <u>rationalisation</u> method: Given $\lim_{x\to p} f(x)$ where $f(x)$ is a quasirational function we rationalise the numerator and/or the denominator, then try to factor the resulting polynomial expressions, cancel any common factors, and try direct substitution again.

Here we use the fact that

$$(\sqrt{A}-B)(\sqrt{A}+B) = A - B^2$$

eg $$\lim_{x\to 0} \frac{1-\sqrt{1+3x}}{x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{1-\sqrt{1+3x}}{x} \cdot \frac{1+\sqrt{1+3x}}{1+\sqrt{1+3x}}$$

$$= \lim_{x\to 0} \frac{1-(1+3x)}{x(1+\sqrt{1+3x})}$$

$$= \lim_{x\to 0} \frac{-3x}{x(1+\sqrt{1+3x})}$$

$$= \lim_{x \to 0} \frac{-3}{1+\sqrt{1+3x}}$$

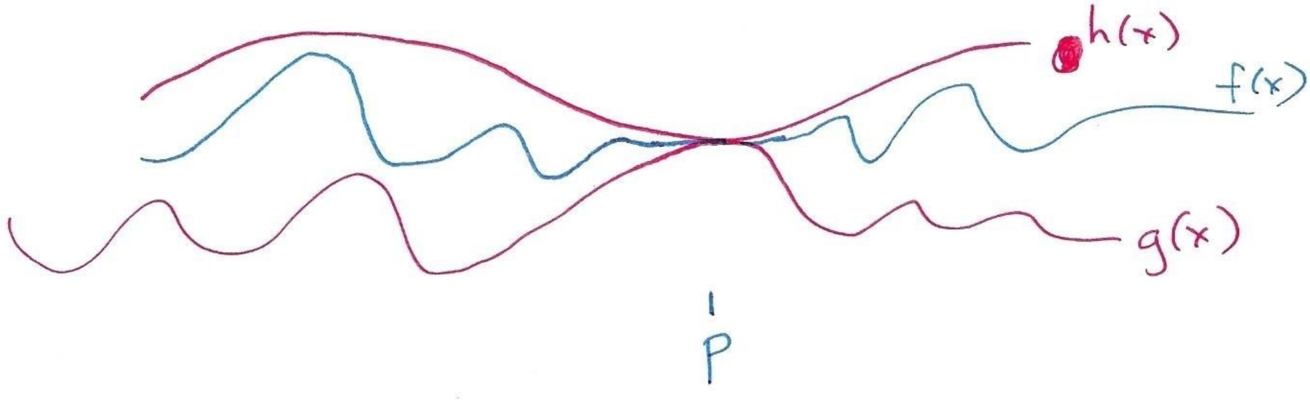
$$= \frac{-3}{2} \quad \boxed{= -\frac{3}{2}}$$

The Squeeze Theorem

Suppose we wish to evaluate $\lim_{x \to p} f(x)$. Further suppose that there exist functions $g(x)$ and $h(x)$ for which $\lim_{x \to p} g(x) = \lim_{x \to p} h(x) = L$ and

$g(x) \leq f(x) \leq h(x)$ for all x near $x = p$.

Then $\lim_{x \to p} f(x) = L$ as well.



eg $\lim_{x \to 0} x^2 \cos\left(\frac{\pi}{2x}\right)$

We <u>cannot</u> write this limit as

$$\left[\lim_{x \to 0} x^2\right] \cdot \left[\lim_{x \to 0} \cos\left(\frac{\pi}{2x}\right)\right]$$

because $\lim_{x \to 0} \cos\left(\frac{\pi}{2x}\right)$ does not exist, so the Basic Limit Properties do <u>not</u> apply.

Recall that

$$-1 \le \cos\left(\frac{\pi}{2x}\right) \le 1$$

$$-x^2 \le x^2 \cos\left(\frac{\pi}{2x}\right) \le x^2$$

Furthermore, $\lim\limits_{x \to 0} x^2 = 0$

$$\lim_{x \to 0} (-x^2) = 0$$

Hence, by the Squeeze Thm, $\lim\limits_{x \to 0} x^2 \cos\left(\frac{\pi}{2x}\right) = 0$.

We can use the Squeeze Thm to prove the following:

Theorem: The Special Trigonometric Limits

1. $\lim\limits_{x \to 0} \frac{\sin(x)}{x} = 1$

2. $\lim\limits_{x \to 0} \frac{\cos(x) - 1}{x} = 0$

Note that 1 can be applied to

$$\lim_{x \to 0} \frac{x}{\sin(x)} = \lim_{x \to 0} \frac{1}{\sin(x)/x}$$

$$= \frac{\lim\limits_{x \to 0} 1}{\lim\limits_{x \to 0} \frac{\sin(x)}{x}}$$

$$= \frac{1}{1}$$

$$\boxed{= 1}$$

Likewise, for ②, we note that

$$\lim_{x\to 0} \frac{1-\cos(x)}{x} = -\lim_{x\to 0} \frac{\cos(x)-1}{x} = -1\cdot 0 \boxed{=0}$$

eg $$\lim_{x\to 0} \frac{\sin(x)}{6x} = \frac{1}{6}\cdot \lim_{x\to 0} \frac{\sin(x)}{x}$$

$$= \frac{1}{6}\cdot 1$$

$$\boxed{= \frac{1}{6}}$$

eg $$\lim_{x\to 0} \frac{\sin(6x)}{x} \cdot \frac{6}{6}$$

$$= 6\cdot \lim_{x\to 0} \frac{\sin(6x)}{6x}$$ As $x\to 0$, $6x \to 0$

$$= 6\cdot 1$$

$$\boxed{= 6}$$

For more difficult limits, and especially those for which other indeterminate forms arise, it is often useful to try rewriting the function in a more convenient form.

eg $\lim_{x \to 1} \left(\frac{1}{x-1} - \frac{2}{x^2-1} \right)$ ($\infty - \infty$ form)

We will try rewriting this as a single rational function by finding the common denominator.

We write

$$\frac{1}{x-1} - \frac{2}{x^2-1} = \frac{1}{x-1} - \frac{2}{(x-1)(x+1)}$$

$$= \frac{1}{x-1} \cdot \frac{x+1}{x+1} - \frac{2}{(x-1)(x+1)}$$

$$= \frac{x+1}{(x-1)(x+1)} - \frac{2}{(x-1)(x+1)}$$

$$= \frac{x-1}{(x-1)(x+1)}$$

$$= \frac{1}{x+1}$$

Hence $\lim_{x\to 1} \left(\frac{1}{x-1} - \frac{2}{x^2-1} \right) = \lim_{x\to 1} \frac{1}{x+1}$

$$\boxed{= \frac{1}{2}}$$