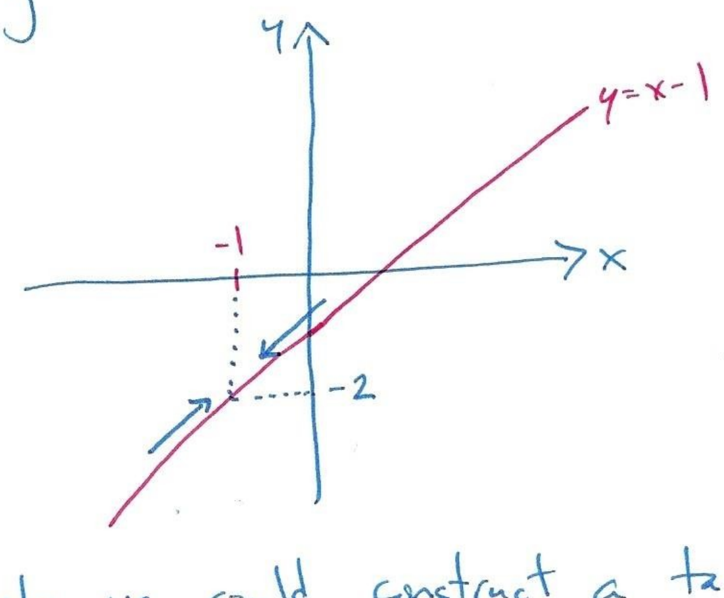
# Section 1.2: Finite Limits

<u>Def'n</u>: Let $p$ and $L$ be real numbers. If a function $f(x)$ becomes arbitrarily close to $L$ as $x$ approaches $p$ from each side, then the <u>limit</u> of $f(x)$ as $x$ tends to $p$ is $L$, and we write $\lim_{x \to p} f(x) = L$.

Specifically, this defines a <u>finite limit</u> because we are requiring $p$ to be a real number.

The easiest way to calculate a limit is by sketching a graph of the function.

eg Let $f(x) = x - 1$. Compute $\lim_{x \to -1} (x-1)$.




Graphically, we can see that $\lim_{x \to -1} (x-1) = -2$.

Alternatively, we could construct a table of values using values approaching $-1$ from either side, and try to determine a pattern in the corresponding values of $f(x)$. This is a <u>numerical</u> approach.

From the left:

| $x$ | $-2$ | $-1.5$ | $-1.1$ | $-1.01$ | $-1.001$ | |
|---|---|---|---|---|---|---|
| $f(x)=x-1$ | $-3$ | $-2.5$ | $-2.1$ | $-2.01$ | $-2.001$ | $\to -2$ |

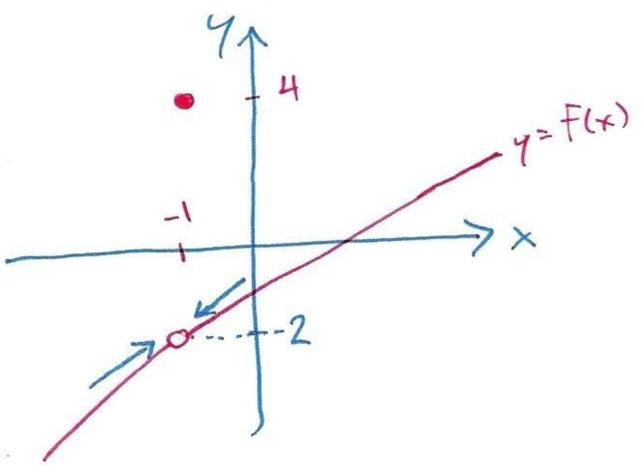
From the right:

| $x$ | $0$ | $-0.5$ | $-0.9$ | $-0.99$ | $-0.999$ | |
|---|---|---|---|---|---|---|
| $f(x)=x-1$ | $-1$ | $-1.5$ | $-1.9$ | $-1.99$ | $-1.999$ | $\to -2$ |

Again, we conclude that $\lim\limits_{x\to -1}(x-1) = -2$.

In this case, $\lim\limits_{x\to -1} f(x) = f(-1)$ which means that the limit can be obtained by <u>direct substitution</u>.

eg Let $f(x) = \begin{cases} x-1, & \text{for } x \neq -1 \\ 4, & \text{for } x = -1 \end{cases}$. Compute $\lim\limits_{x\to -1} f(x)$.



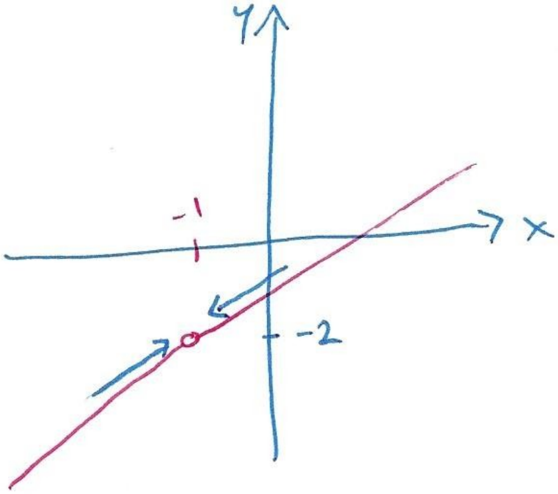

Graphically, we see that $\lim\limits_{x\to -1} f(x) = -2$.

However, now $\lim\limits_{x\to -1} f(x) \neq f(-1)$.

eg Let $f(x) = \dfrac{x^2-1}{x+1}$. Compute $\lim\limits_{x\to -1} f(x)$.

Observe that $f(x) = \dfrac{x^2-1}{x+1} = \dfrac{(x-1)(x+1)}{x+1}$

$$= \begin{cases} x-1, & \text{for } x \neq -1 \\ \text{undefined}, & \text{for } x = -1 \end{cases}$$




Graphically, we see that

$$\lim_{x \to -1} f(x) = -2,$$

despite the fact that $f(-1)$ is undefined.

In general, a function may be defined or undefined at a point $x = p$, and $\lim_{x \to p} f(x)$ may exist or may not exist.

eg $\displaystyle\lim_{x \to 0} \frac{\sin(x)}{x}$

| $x$ | $\pm 1$ | $\pm 0.5$ | $\pm 0.1$ | $\pm 0.05$ | $\pm 0.01$ |
|---|---|---|---|---|---|
| $\frac{\sin(x)}{x}$ | 0.8415 | 0.9589 | 0.9983 | 0.9996 | 0.99998 $\to 1$ |

Numerically, we see that $\displaystyle\lim_{x \to 0} \frac{\sin(x)}{x} = 1$.

Unfortunately, it is not hard to draw <u>incorrect</u> conclusions using the numerical method for limits.

eg $\lim_{x \to 0} \left(x^2 + \frac{2^x}{1000}\right)$

From the left:

| $x$ | $-1$ | $-0.5$ | $-0.1$ | $-0.05$ | $-0.01$ |
|---|---|---|---|---|---|
| $f(x)$ | 1.0005 | 0.25071 | 0.01093 | 0.00347 | 0.00109 $\to 0$ |

From the right:

| $x$ | $1$ | $0.5$ | $0.1$ | $0.05$ | $0.01$ |
|---|---|---|---|---|---|
| $f(x)$ | 1.002 | 0.25141 | 0.01107 | 0.00354 | 0.00111 $\to 0$ |

We conclude that $\lim_{x \to 0} \left(x^2 + \frac{2^x}{1000}\right) = 0$.

Unfortunately, this is <u>wrong</u>! Suppose we extend the first table of values:

| $x$ | $-0.005$ | $-0.001$ | $-0.0005$ |
|---|---|---|---|
| $f(x)$ | 0.00102 | 0.0010003 | 0.0010000... $\to 0.001 = \frac{1}{1000}$ |

Likewise, from the right:

| $x$ | $0.005$ | $0.001$ | $0.0005$ |
|---|---|---|---|
| $f(x)$ | 0.00103 | 0.001002 | 0.0010006 $\to 0.001$ |

$$\boxed{\text{Thus } \lim_{x \to 0} \left(x^2 + \frac{2^x}{1000}\right) = \frac{1}{1000}}$$

eg $\lim_{x \to 0} \cos\left(\frac{\pi}{2x}\right)$

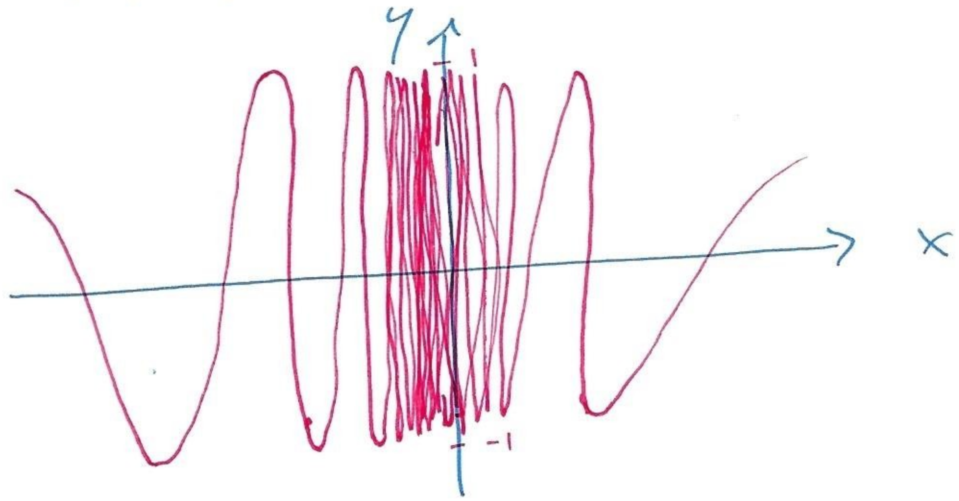
| $x$ | $\pm 0.05$ | $\pm 0.01$ | $\pm 0.005$ | $\pm 0.001$ | $\pm 0.0005$ | |
|---|---|---|---|---|---|---|
| $\cos\left(\frac{\pi}{2x}\right)$ | 1 | 1 | 1 | 1 | 1 | $\to 1$ |

It appears that $\lim_{x \to 0} \cos\left(\frac{\pi}{2x}\right) = 1$.

Unfortunately, this is <u>wrong</u>! To see this, let's use a different set of values to approach $x=0$.

| $x$ | $\pm 0.02$ | $\pm 0.004$ | $\pm 0.0008$ | |
|---|---|---|---|---|
| $\cos\left(\frac{\pi}{2x}\right)$ | $-1$ | $-1$ | $-1$ | $\to -1$ |

This suggests that $\cos\left(\frac{\pi}{2x}\right)$ is oscillating between 1 and $-1$ an infinite number of times as $x \to 0$.




Thus we say that $\lim_{x \to 0} \cos\left(\frac{\pi}{2x}\right)$ does not exist.

Def'n: Let $p$ and $L_\ell$ be real numbers. If a function $f(x)$ becomes arbitrarily close to $L_\ell$ as $x$ approaches $p$ from the left, then the lefthand limit of $f(x)$ as $x$ tends to $p$ is $L_\ell$, and we write $\lim_{x \to p^-} f(x) = L_\ell$.

Def'n: Let $p$ and $L_r$ be real numbers. If a function $f(x)$ becomes arbitrarily close to $L_r$ as $x$ approaches $p$ from the right, then the righthand limit of $f(x)$ as $x$ tends to $p$ is $L_r$, and we write $\lim_{x \to p^+} f(x) = L_r$.

Collectively, these are known as one-sided limits.

Theorem: $\lim_{x \to p} f(x) = L$ if and only if

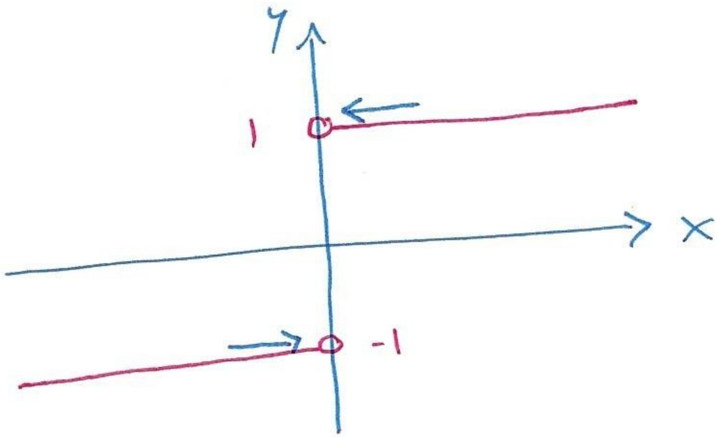
$$\lim_{x \to p^-} f(x) = L \text{ and } \lim_{x \to p^+} f(x) = L.$$

eg $\lim_{x \to 0} \frac{|x|}{x}$

Recall that

$$|x| = \begin{cases} x, & \text{for } x \geq 0 \\ -x, & \text{for } x < 0. \end{cases}$$

Thus $$\frac{|x|}{x} = \begin{cases} \frac{x}{x} = 1, & \text{for } x > 0 \\ \frac{-x}{x} = -1, & \text{for } x < 0 \end{cases}$$




Graphically, we see that $\lim_{x \to 0^-} \frac{|x|}{x} = -1$.

However, $\lim_{x \to 0^+} \frac{|x|}{x} = 1$.

Because the one-sided limits differ,

$\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

eg $\lim_{x \to 1} \frac{1}{(x-1)^2}$

| $x$ | 0.5 | 0.9 | 0.99 | 0.999 | |
|---|---|---|---|---|---|
| $\frac{1}{(x-1)^2}$ | 4 | 100 | 10000 | 1000000 | $\to \infty$ |

When a function becomes <u>unboundedly large</u> as $x \to p$, the limit does <u>not</u> exist but we can assign $\infty$ to describe how the limit does not exist. Here,

$$\lim_{x \to 1^-} \frac{1}{(x-1)^2} = \infty.$$

| $x$ | 1.5 | 1.1 | 1.01 | 1.001 | |
|---|---|---|---|---|---|
| $\frac{1}{(x-1)^2}$ | 4 | 100 | 10000 | 1000000 | $\to \infty$ |

Likewise, $\lim_{x \to 1^+} \frac{1}{(x-1)^2} = \infty$.

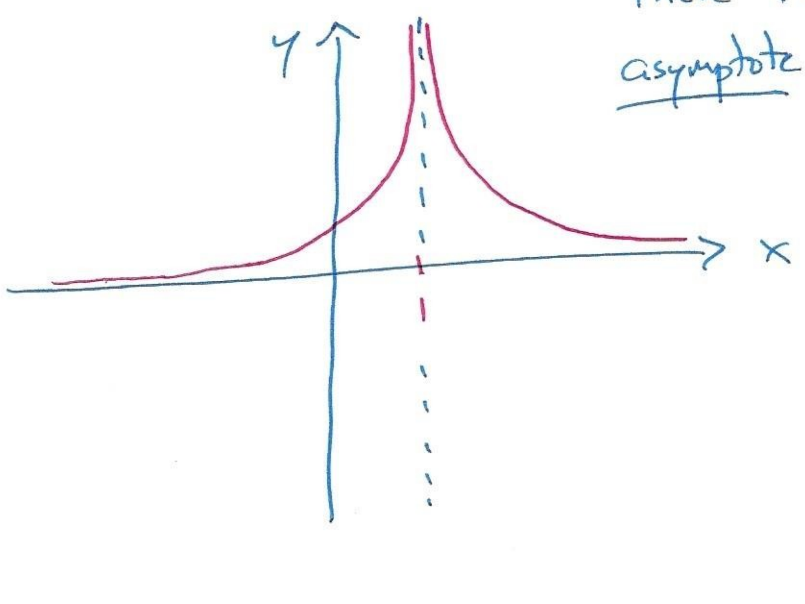
Because the one-sided limits agree, we can assign $\lim_{x \to 1} \frac{1}{(x-1)^2} = \infty$ even though this means that the limit does <u>not</u> exist.

If a function $f(x)$ attains values that become unboundedly large as $x \to p^-$ or as $x \to p^+$ then $\lim_{x \to p} f(x)$ does not exist. If the values are positive then we assign $\infty$ to that one-sided limit. If the values are negative then we assign $-\infty$. If the behaviour is the same for both one-sided limits then we can assign $\infty$ or $-\infty$ to $\lim_{x \to p} f(x)$. This is called an <u>infinite limit</u>.

Infinite limits have an important graphical connection.

eg $\lim_{x \to 1} \frac{1}{(x-1)^2}$

There is a <u>vertical asymptote</u> at $x=1$.



If $\lim_{x \to p^-} f(x)$ or $\lim_{x \to p^+} f(x)$ is an infinite limit then the line $x = p$ must be a vertical asymptote to the graph of $y = f(x)$.

In general, if direct substitution of $x = p$ into $f(x)$ produces a $\frac{K}{0}$ form for $K \neq 0$ then the corresponding one-sided limits will be infinite, and $x = p$ will be a vertical asymptote.

eg $f(x) = \dfrac{x+6}{2-x}$

We set $2 - x = 0$

$x = 2$

Since direct substitution gives $f(2) = \frac{8}{0}$, this is a $\frac{K}{0}$ form and so $x = 2$ must be a vertical asymptote.

From the left as $x \to 2$, $2 - x$ is a small positive number and hence $\dfrac{x+6}{2-x}$ is a large positive number.

This means $\displaystyle\lim_{x \to 2^-} \frac{x+6}{2-x} = \infty$.

From the right as $x \to 2$, $2-x$ is a small negative number and hence

$\frac{x+6}{2-x}$ is a large negative number.

Thus $\lim\limits_{x \to 2^+} \frac{x+6}{2-x} = -\infty$.

Since the one-sided limits disagree, we cannot assign $\infty$ or $-\infty$ to $\lim\limits_{x \to 2} \frac{x+6}{2-x}$.

Unfortunately, if direct substitution yields a $\frac{0}{0}$ form, we cannot conclude that this is an infinite limit. This is an indeterminate form.