MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Assignment 8	**MATHEMATICS 1000**	Fall 2024

SOLUTIONS

[4] 1. We begin by finding the first derivative:

$$\begin{aligned}\frac{d}{dx}[y] &= \frac{d}{dx}[2x - y^2]\\ \frac{dy}{dx} &= 2 - 2y\frac{dy}{dx}\\ \frac{dy}{dx} + 2y\frac{dy}{dx} &= 2\\ \frac{dy}{dx}(1+2y) &= 2\\ \frac{dy}{dx} &= \frac{2}{1+2y}.\end{aligned}$$

Now we differentiate again:

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left[\frac{2}{1+2y}\right]\\ &= \frac{0 - 2\cdot 2\frac{dy}{dx}}{(1+2y)^2}\\ &= \frac{-4\frac{dy}{dx}}{(1+2y)^2}\\ &= \frac{-4\left(\frac{2}{1+2y}\right)}{(1+2y)^2}\\ &= \frac{-8}{(1+2y)^3}.\end{aligned}$$

[5] 2. Let V be the volume of the cone, r be its radius and h be its height. We know that r is constant and $\frac{dh}{dt} = -\frac{1}{3}$, and we want to find $\frac{dV}{dt}$ when $h = 1$ and $V = 3\pi$. Then we have

$$\begin{aligned}V &= \frac{\pi}{3}r^2h\\ \frac{dV}{dt} &= \frac{\pi}{3}r^2\frac{dh}{dt}.\end{aligned}$$

From the original equation, when $h = 1$ and $V = 3\pi$, $r = 3$ so then

$$\frac{dV}{dt} = \frac{\pi}{3}(3^2)\left(-\frac{1}{3}\right) = -\pi$$

and therefore the cone's volume is shrinking by π cubic metres per hour.

[5] 3. Suppose that P is the point on the stage closest to the spotlight. Let x be the distance from the woman to P, y be the distance from the spotlight to P, ℓ be the distance from the spotlight to the woman, and θ be the angle be the lines representing the last two distances, as in Figure 1. We are given that $y = 60$ and, if we assume that the woman is walking away from P, then $\frac{dx}{dt} = 3$. (We could alternatively set up the problem by assuming that the woman is walking towards P. In the case, $\frac{dx}{dt} = -3$, but the details of the solution remain the same.) We want to find $\frac{d\theta}{dt}$ at the moment when $x = 25$.

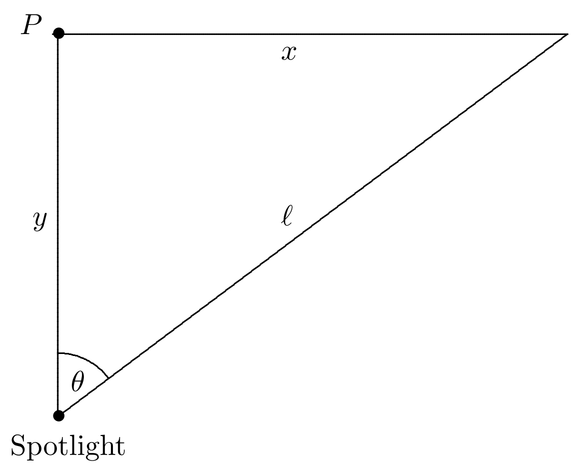

Figure 1: A spotlight shines on a woman walking across a stage.

From trigonometry, we can see that

$$\begin{aligned} \tan(\theta) &= \frac{x}{y} \\ \frac{d}{dt}[\tan(\theta)] &= \frac{d}{dt}\left[\frac{x}{y}\right] \\ \sec^2(\theta)\frac{d\theta}{dt} &= \frac{1}{y}\cdot\frac{dx}{dt}, \end{aligned}$$

where we have used the fact that y is a constant. In order to determine $\sec^2(\theta)$ at the given moment, we can use the Pythagorean theorem to conclude that when $x = 25$ and $y = 60$,

$$\ell = \sqrt{25^2 + 60^2} = \sqrt{4225} = 65.$$

Thus $\cos(\theta) = \frac{60}{65} = \frac{12}{13}$ and so $\sec^2(\theta) = \left(\frac{13}{12}\right)^2 = \frac{169}{144}$. Now we see that

$$\begin{aligned} \frac{169}{144}\cdot\frac{d\theta}{dt} &= \frac{1}{60}\cdot 3 \\ \frac{d\theta}{dt} &= \frac{36}{845}, \end{aligned}$$

and so the spotlight is rotating at a rate of $\frac{36}{845}$ radians per second. (If we assumed from the start that the woman was walking away from P, the rate would be negative.)

[3] 4. (a) We need to find any critical points. Note that $f'(x)$ is undefined only when $x = 1$, which is not in the domain of $f(x)$, so we need only consider $f'(x) = 0$. Thus we set $-x(5x + 4) = 0$ so $x = 0$ or $x = -\frac{4}{5}$. We can now construct the sign pattern found in Figure 2. We can see that $f(x)$ is increasing for $-\frac{4}{5} < x < 0$ and decreasing for $x < -\frac{4}{5}$, $0 < x < 1$ and $x > 1$. Furthermore, we have a relative minimum at $x = -\frac{4}{5}$. We have a relative maximum at $x = 0$.

Figure 2: Sign patterns for Question 6(a).

[3] (b) We find the hypercritical points. Again, $f''(x)$ is undefined only at $x = 1$, which is not in the domain of $f(x)$. Furthermore, $f''(x) = 0$ when $2(5x+1)(x+2) = 0$, that is when $x = -\frac{1}{5}$ or $x = -2$. We therefore construct the sign pattern found in Figure 2. We conclude that $f(x)$ is concave upward for $-2 < x < -\frac{1}{5}$ and $x > 1$ and concave downward for $x < -2$ and $-\frac{1}{5} < x < 1$. The points of inflection occur at $x = -2$ and $x = -\frac{1}{5}$.