MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Assignment 7 | **MATHEMATICS 1000** | Fall 2022

SOLUTIONS

[4] 1. First observe that

$$f'(x) = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot [\sqrt{x}]' = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x-x^2}}.$$

Thus the slope of the tangent line is

$$f'\left(\frac{1}{2}\right) = \frac{1}{2\sqrt{\frac{1}{2}-\frac{1}{4}}} = \frac{1}{2\cdot\frac{1}{2}} = 1.$$

Additionally, the y-coordinate when $x = \frac{1}{2}$ is

$$f\left(\frac{1}{2}\right) = \arcsin\left(\sqrt{\frac{1}{2}}\right) = \arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}.$$

Hence the equation of the tangent line is

$$y - \frac{\pi}{4} = 1\cdot\left(x-\frac{1}{2}\right) \implies \boxed{y = x + \frac{\pi}{4} - \frac{1}{2}}.$$

[4] 2. (a) First we have

$$y' = \cosh(\arctan(x)) \cdot [\arctan(x)]' = \cosh(\arctan(x)) \cdot \frac{1}{x^2+1} = \frac{\cosh(\arctan(x))}{x^2+1}.$$

Next,

$$y'' = \frac{[\cosh(\arctan(x))]' \cdot (x^2+1) - \cosh(\arctan(x)) \cdot [x^2+1]'}{(x^2+1)^2}$$
$$= \frac{\sinh(\arctan(x)) \cdot [\arctan(x)]' \cdot (x^2+1) - \cosh(\arctan(x)) \cdot 2x}{(x^2+1)^2}$$
$$= \frac{\sinh(\arctan(x)) \cdot \frac{1}{x^2+1} \cdot (x^2+1) - 2x\cosh(\arctan(x))}{(x^2+1)^2}$$
$$\boxed{= \frac{\sinh(\arctan(x)) - 2x\cosh(\arctan(x))}{(x^2+1)^2}}.$$

[4] (b) We differentiate implicitly:

$$\begin{aligned}\frac{d}{dx}\left[\sqrt{x}-\sqrt{y}\right] &= \frac{d}{dx}[4]\\ \frac{1}{2\sqrt{x}}-\frac{1}{2\sqrt{y}}\cdot\frac{dy}{dx} &= 0\\ \frac{dy}{dx} &= \frac{\sqrt{y}}{\sqrt{x}}=\sqrt{\frac{y}{x}}.\end{aligned}$$

Now we differentiate implicitly a second time:

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{\frac{1}{2\sqrt{y}}\cdot\frac{dy}{dx}\cdot\sqrt{x}-\frac{1}{2\sqrt{x}}\cdot\sqrt{y}}{x}\\ &= \frac{\frac{1}{2\sqrt{y}}\cdot\frac{\sqrt{y}}{\sqrt{x}}\cdot\sqrt{x}-\frac{1}{2\sqrt{x}}\cdot\sqrt{y}}{x}\\ &= \frac{\frac{1}{2}-\frac{1}{2\sqrt{x}}\cdot\sqrt{y}}{x}\\ &= \frac{\sqrt{x}-\sqrt{y}}{2x^{\frac{3}{2}}}.\end{aligned}$$

Because the equation here is $\sqrt{x}-\sqrt{y}=4$, we can further simplify this expression to

$$\frac{d^2y}{dx^2}=\frac{4}{2x^{\frac{3}{2}}}=\frac{2}{x^{\frac{3}{2}}}.$$

[4] 3. Let x be the distance between Guerrero and second base, and ℓ be his distance to third base. We are given that $\frac{dx}{dt}=-27$ (because x will shrink as Donaldson approaches second base). We want to find $\frac{d\ell}{dt}$ when $x=45$. Observe that we have a right triangle joining Guerrero, second base and third base; this right triangle has a hypotenuse of length ℓ and sides of length x and 90. By the Pythagorean Theorem, then,

$$x^2+90^2=\ell^2.$$

Thus

$$\begin{aligned}\frac{d}{dt}[x^2+90^2] &= \frac{d}{dt}[\ell^2]\\ 2x\frac{dx}{dt}+0 &= 2\ell\frac{d\ell}{dt}\\ x\frac{dx}{dt} &= \ell\frac{d\ell}{dt}.\end{aligned}$$

When Guerrero is halfway to second base,

$$45^2+90^2=\ell^2 \quad\Longrightarrow\quad \ell=45\sqrt{5}.$$

Thus

$$45(-27) = 45\sqrt{5}\frac{d\ell}{dt}$$
$$\frac{d\ell}{dt} = -\frac{27}{\sqrt{5}} = -\frac{27\sqrt{5}}{5} \approx -12.1.$$

Thus Guerrero's distance to third base is decreasing at a rate of about 12.1 feet per second.

[4] 4. Let h be the height of the child's shadow and x be the distance from the child to the spotlight. Let H be the child's height, and X be the distance from the spotlight to the wall. We are given that $H = 1$ and $X = 10$. Furthermore, the information about the rate at which the height of the shadow shrinks can be converted to metres per second, which means that $\frac{dh}{dt} = -12.5 \cdot \frac{1}{100} = -\frac{1}{8}$. We want to find $\frac{dx}{dt}$ when $x = 4$ (because if she is 6 metres from the wall then $x = 10 - 6 = 4$). As depicted in Figure 1, we are dealing with two right triangles which each contain the angle that the beam of light makes with the ground. Thus these two triangles are similar.

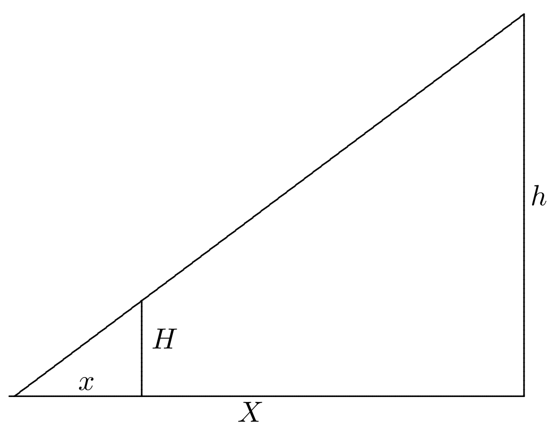

Figure 1: A child walks away from a spotlight.

Since the corresponding sides of similar triangles are always porportional, we have

$$\begin{aligned}
\frac{h}{H} &= \frac{X}{x} \\
\frac{d}{dt}\left[\frac{h}{H}\right] &= \frac{d}{dt}\left[\frac{X}{x}\right] \\
\frac{1}{H} \cdot \frac{dh}{dt} &= -\frac{X}{x^2} \cdot \frac{dx}{dt} \\
\frac{1}{1} \cdot \left(-\frac{1}{8}\right) &= -\frac{10}{4^2} \cdot \frac{dx}{dt} \\
\frac{dx}{dt} &= \frac{1}{5}.
\end{aligned}$$

We conclude that the child is walking forwards at a rate of $\frac{1}{5}$ metres per second.