MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 4 | **MATHEMATICS 1000** | FALL 2024

SOLUTIONS

[5] 1. We have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{x+h+1}{x+h-3} - \frac{x+1}{x-3}}{h} \\
&= \lim_{h\to 0} \left(\frac{(x+h+1)(x-3)-(x+1)(x+h-3)}{(x-3)(x+h-3)} \right) \cdot \frac{1}{h} \\
&= \lim_{h\to 0} \frac{(x^2+hx+x-3x-3h-3)-(x^2+hx-3x+x+h-3)}{h(x-3)(x+h-3)} \\
&= \lim_{h\to 0} \frac{-4h}{h(x-3)(x+h-3)} \\
&= \lim_{h\to 0} \frac{-4}{(x-3)(x+h-3)} \\
&= \frac{-4}{(x-3)^2}.
\end{aligned}$$

[5] 2. (a) We have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{5(x+h)-1} - \sqrt{5x-1}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{5x+5h-1} - \sqrt{5x-1}}{h} \cdot \frac{\sqrt{5x+5h-1} + \sqrt{5x-1}}{\sqrt{5x+5h-1} + \sqrt{5x-1}} \\
&= \lim_{h\to 0} \frac{(5x+5h-1)-(5x-1)}{h\left[\sqrt{5x+5h-1} + \sqrt{5x-1}\right]} \\
&= \lim_{h\to 0} \frac{5h}{h\left[\sqrt{5x+5h-1} + \sqrt{5x-1}\right]} \\
&= \lim_{h\to 0} \frac{5}{\left[\sqrt{5x+5h-1} + \sqrt{5x-1}\right]} \\
&= \frac{5}{2\sqrt{5x-1}}.
\end{aligned}$$

[2] (b) From part (a), the slope of the tangent line at $x = 2$ is $f'(2) = \frac{5}{6}$. Furthermore, the y-coordinate of the point at $x = 2$ is $f(2) = 3$. Thus the tangent line has the form

$$y = \frac{5}{6}x + b$$

where

$$\begin{aligned} 3 &= \frac{5}{6} \cdot 2 + b \\ \frac{4}{3} &= b. \end{aligned}$$

The equation of the tangent line is therefore $y = \frac{5}{6}x + \frac{4}{3}$.

[4] 3. (a) We use the alternative definition of the limit:

$$f'(-4) = \lim_{x \to -4} \frac{f(x) - f(-4)}{x - (-4)} = \lim_{x \to -4} \frac{f(x) - 9}{x + 4}.$$

From the left,

$$\begin{aligned} \lim_{x \to -4^-} \frac{f(x) - 9}{x + 4} &= \lim_{x \to -4^-} \frac{(2x^2 + 8x + 9) - 9}{x + 4} \\ &= \lim_{x \to -4^-} \frac{2x^2 + 8x}{x + 4} \\ &= \lim_{x \to -4^-} \frac{2x(x + 4)}{x + 4} \\ &= \lim_{x \to -4^-} 2x \\ &= -8. \end{aligned}$$

From the right,

$$\begin{aligned} \lim_{x \to -4^+} \frac{f(x) - 9}{x + 4} &= \lim_{x \to -4^+} \frac{(x^2 - 7) - 9}{x + 4} \\ &= \lim_{x \to -4^+} \frac{x^2 - 16}{x + 4} \\ &= \lim_{x \to -4^+} \frac{(x - 4)(x + 4)}{x + 4} \\ &= \lim_{x \to -4^+} (x - 4) \\ &= -8. \end{aligned}$$

Since the one-sided limits agree, the limit exists and therefore $f(x)$ is differentiable at $x = -4$.

[4] (b) This time, the alternative definition of the limit gives

$$g'(-4) = \lim_{x \to -4} \frac{g(x) - g(-4)}{x - (-4)} = \lim_{x \to -4} \frac{g(x) - 9}{x + 4}.$$

From the left, we have

$$\begin{aligned} \lim_{x \to -4^-} \frac{g(x) - 9}{x + 4} &= \lim_{x \to -4^-} \frac{(1 - 2x) - 9}{x + 4} \\ &= \lim_{x \to -4^-} \frac{-2x - 8}{x + 4} \\ &= \lim_{x \to -4^-} \frac{-2(x + 4)}{x + 4} \\ &= \lim_{x \to -4^-} -2 \\ &= -2. \end{aligned}$$

From the right, however, we again have

$$\begin{aligned} \lim_{x \to -4^+} \frac{g(x) - 9}{x + 4} &= \lim_{x \to -4^+} \frac{(x^2 - 7) - 9}{x + 4} \\ &= \lim_{x \to -4^+} \frac{x^2 - 16}{x + 4} \\ &= \lim_{x \to -4^+} \frac{(x - 4)(x + 4)}{x + 4} \\ &= \lim_{x \to -4^+} (x - 4) \\ &= -8. \end{aligned}$$

Since the one-sided limits differ, the limit does not exist and therefore $g(x)$ is not differentiable at $x = -4$.