MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 4.3 **Math 1000 Worksheet** FALL 2022

SOLUTIONS

1. (a) First we observe that $f(x)$ is defined except when $(x-1)^3 = 0$ so $x = 1$. Note that the numerator is non-zero, so $x = 1$ is a vertical asymptote.

To find any horizontal asymptotes, we evaluate

$$\begin{aligned}\lim_{x\to\infty} \frac{x^2(x+2)}{(x-1)^3} &= \lim_{x\to\infty} \frac{x^3+2x^2}{x^3-3x^2+3x-1} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} \\ &= \lim_{x\to\infty} \frac{1+\frac{2}{x}}{1-\frac{3}{x}+\frac{3}{x^2}-\frac{1}{x^3}} \\ &= \frac{1+0}{1-0+0-0} \\ &= 1.\end{aligned}$$

Thus $y = 1$ is a horizontal asymptote, and since $f(x)$ is a rational function, it must approach the horizontal asymptote both as $x \to \infty$ and as $x \to -\infty$.

To find any x-intercepts, we set $f(x) = 0$, so $x^2(x+2) = 0$, so $x = 0$ or $x = -2$. Thus the points $(0,0)$ and $(-2,0)$ are the x-intercepts. This also means that $(0,0)$ is the y-intercept, which we could alternatively find by evaluating $f(0)$.

Now we need to find any critical points. Note that $f'(x)$ is undefined only when $x = 1$ (the vertical asymptote) so we need only consider $f'(x) = 0$, that is, $-x(5x+4) = 0$, so $x = 0$ or $x = -\frac{4}{5}$. We can now construct the sign pattern found in Figure 1. We can see that $f(x)$ is increasing on the interval $-\frac{4}{5} < x < 0$ and decreasing on the intervals $x < -\frac{4}{5}$, $0 < x < 1$ and $x > 1$. Furthermore, we have a relative minimum at $x = -\frac{4}{5}$, which is the point $\left(-\frac{4}{5}, -\frac{32}{243}\right)$. We have a relative maximum at $x = 0$, which is the point $(0,0)$.

Figure 1: Sign patterns for Question 1(a).

Finally, we find the hypercritical points. Again, $f''(x)$ is undefined only at the vertical asymptote $x = 1$. Furthermore, $f''(x) = 0$ when $2(5x+1)(x+2) = 0$, that is when $x = -\frac{1}{5}$ or $x = -2$. We therefore construct the sign pattern found in Figure 1. We conclude that $f(x)$ is concave upward on the intervals $-2 < x < -\frac{1}{5}$ and $x > 1$ and concave downward on the intervals $x < -2$ and $-\frac{1}{5} < x < 1$. The points of inflection occur at $x = -2$ and $x = -\frac{1}{5}$, which are the points $(-2, 0)$ and $\left(-\frac{1}{5}, -\frac{1}{24}\right)$.

Now we can sketch the graph of $f(x)$, as found in Figure 2.

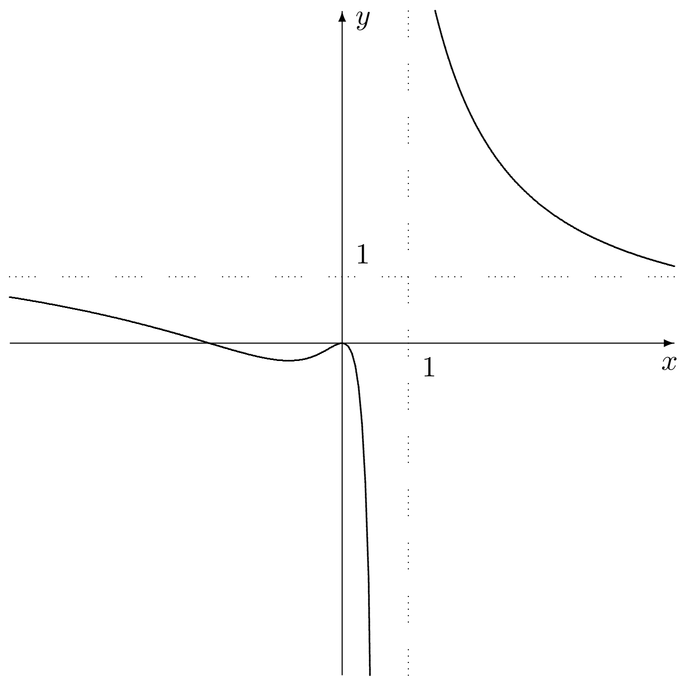

Figure 2: The graph for Question 1(a).

(b) Observe that $f(x)$ is defined anywhere because if $x^2 + 1 = 0$ then $x^2 = -1$, which has no solutions. Thus there are no vertical asymptotes or other discontinuities. For the horizontal asymptotes, we observe that $f(x)$ is rational so we need only take one limit at infinity:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x^2 - 4x + 2}{x^2 + 1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{2 - \frac{4}{x} + \frac{2}{x^2}}{1 + \frac{1}{x^2}} = \frac{2 - 0 + 0}{1 + 0} = 2,$$

giving $y = 2$ as the horizontal asymptote.

Now we seek the x-intercepts. Setting $f(x) = 0$ gives

$$2x^2 - 4x + 2 = 2(x-1)^2 = 0$$

so $x = 1$; hence $(1, 0)$ is the only x-intercept. Next we evaluate $f(0) = 2$, so $(0, 2)$ is the y-intercept.

Now we consider $f'(x)$. Observe that it never fails to exist since the denominator is always positive. Thus we set $f'(x) = 0$ so

$$4x^2 - 4 = 4(x+1)(x-1) = 0$$

giving $x = 1$ and $x = -1$ as the critical points. From the sign pattern given in Figure 3 we can see that $f(x)$ is increasing on $x < -1$ and $x > 1$, while it is decreasing on $-1 < x < 1$. Hence the point $(-1, 4)$ is a relative maximum while the point $(1, 0)$ is a relative minimum.

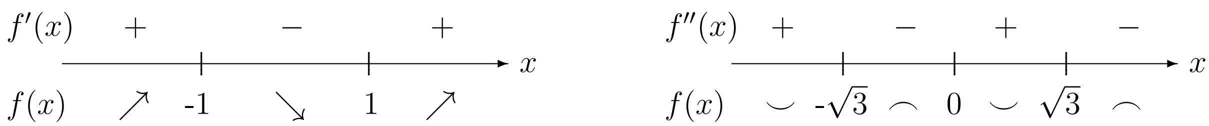

Figure 3: The sign patterns for Question 1(b).

Finally, we consider concavity. Again, $f''(x)$ never fails to exist so we set $f''(x) = 0$, giving

$$24x - 8x^3 = 8x(x^2 - 3) = 0.$$

Thus $x = 0$, $x = \sqrt{3}$ and $x = -\sqrt{3}$ are the hypercritical points. From the sign pattern, we see that $f(x)$ is concave upward on $x < -\sqrt{3}$ and $0 < x < \sqrt{3}$, and it is concave downward on $-\sqrt{3} < x < 0$ and $x > \sqrt{3}$. Hence the points $(-\sqrt{3}, 2 + \sqrt{3})$, $(0, 2)$ and $(\sqrt{3}, 2 - \sqrt{3})$ are all inflection points.

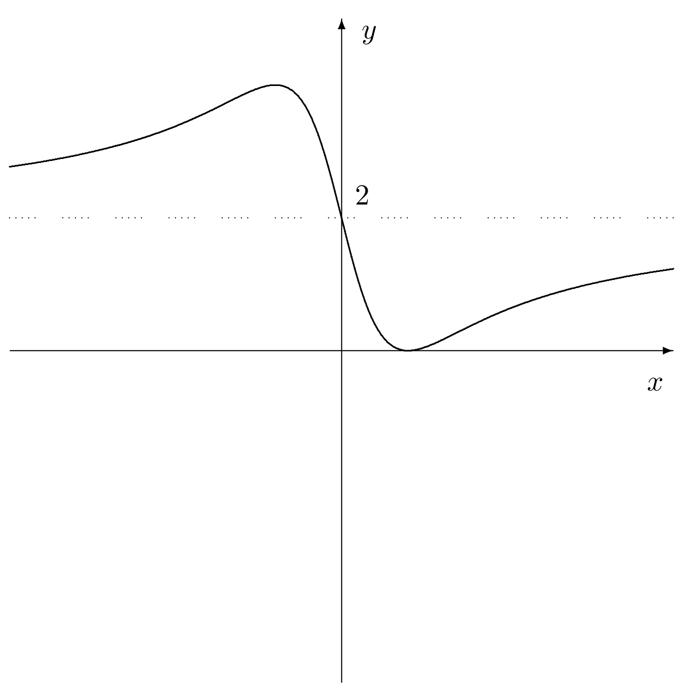

Figure 4: The graph for Question 1(b).

We can now sketch the graph depicted in Figure 4.

(c) The domain of $f(x)$ consists of all real numbers because $x^2 + 1 > 0$ and so the square root can neither have a negative argument nor be equal to zero. Thus there are no vertical asymptotes or other discontinuities. Since this is a quasirational function, $f(x)$ may have as many as two different horizontal asymptotes, so we must check both limits at infinity. First,

$$\lim_{x\to\infty} \frac{x+3}{\sqrt{x^2+1}} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{1+\frac{3}{x}}{\frac{1}{\sqrt{x^2}}\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{1+\frac{3}{x}}{\sqrt{1+\frac{1}{x^2}}} = \frac{1+0}{\sqrt{1+0}} = 1.$$

Thus the graph approaches the horizontal asymptote $y = 1$ as $x \to \infty$. Next,

$$\lim_{x\to-\infty} \frac{x+3}{\sqrt{x^2+1}} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to-\infty} \frac{1+\frac{3}{x}}{\frac{1}{-\sqrt{x^2}}\sqrt{x^2+1}} = \lim_{x\to-\infty} \frac{1+\frac{3}{x}}{-\sqrt{1+\frac{1}{x^2}}} = \frac{1+0}{-\sqrt{1+0}} = -1.$$

This means that the graph approaches the horizontal asymptote $y = -1$ as $x \to -\infty$. Now we check for any x-intercepts. We set $x + 3 = 0$ so $x = -3$, and conclude that the point $(-3, 0)$ is the only x-intercept. Also, $f(0) = 3$ so $(0, 3)$ is the y-intercept.

Next we consider $f'(x)$. It is never undefined, and so we set $1 - 3x = 0$, which means that $x = \frac{1}{3}$ is the only critical point. From the sign pattern given in Figure 5 we can see that $f(x)$ is increasing on $x < \frac{1}{3}$ and decreasing on $x > \frac{1}{3}$. Hence the point $\left(\frac{1}{3}, \sqrt{10}\right)$ is a relative maximum.

Figure 5: Sign patterns for Question 1(c).

Finally, we consider concavity. Again, $f''(x)$ never fails to exist so we set $f''(x) = 0$, giving $3(2x + 1)(x - 1) = 0$ and so $x = -\frac{1}{2}$ and $x = 1$ are the hypercritical points. From the sign pattern, we see that $f(x)$ is concave upward on $x < -\frac{1}{2}$ and $x > 1$, and it is concave downward on $-\frac{1}{2} < x < 1$. Hence the points $\left(-\frac{1}{2}, \sqrt{5}\right)$ and $\left(1, 2\sqrt{2}\right)$ are both inflection points.

We can now sketch the graph depicted in Figure 6.

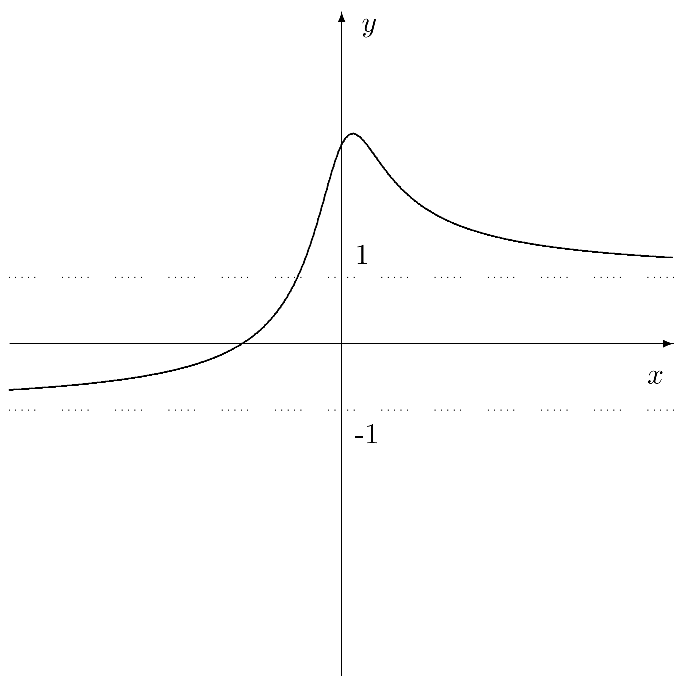

Figure 6: The graph for Question 1(c).