MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

TEST 3 | **MATHEMATICS 1000-002** | FALL 2022

SOLUTIONS

[5] 1. (a) We use the Chain Rule, followed by the Product Rule:

$$\begin{aligned} y' &= \cos(x^2\tan(x)) \cdot [x^2\tan(x)]' \\ &= \cos(x^2\tan(x)) \cdot ([x^2]'\tan(x) + x^2[\tan(x)]') \\ &= \cos(x^2\tan(x)) \cdot (2x\tan(x) + x^2 \cdot \sec^2(x)) \\ &= 2x\tan(x)\cos(2x\tan(x)) + x^2\sec^2(x)\cos(x^2\tan(x)). \end{aligned}$$

[5] (b) We use the Product Rule, followed by the Chain Rule:

$$\begin{aligned} y' &= [x^2]'\sin(\tan(x)) + x^2[\sin(\tan(x))]' \\ &= 2x\sin(\tan(x)) + x^2 \cdot \cos(\tan(x)) \cdot [\tan(x)]' \\ &= 2x\sin(\tan(x)) + x^2\cos(\tan(x))\sec^2(x). \end{aligned}$$

[5] (c) We use the Chain Rule twice:

$$\begin{aligned} y' &= 2\sin(\tan(x)) \cdot [\sin(\tan(x))]' \\ &= 2\sin(\tan(x)) \cdot \cos(\tan(x)) \cdot [\tan(x)]' \\ &= 2\sin(\tan(x))\cos(\tan(x))\sec^2(x). \end{aligned}$$

[5] (d) We use the Quotient Rule, followed by the Chain Rule:

$$\begin{aligned} f'(x) &= \frac{[e^{2x}-1]'(e^{2x}+1) - (e^{2x}-1)[e^{2x}+1]'}{(e^{2x}+1)^2} \\ &= \frac{e^{2x} \cdot [2x]' \cdot (e^{2x}+1) - (e^{2x}-1) \cdot e^{2x} \cdot [2x]'}{(e^{2x}+1)^2} \\ &= \frac{e^{2x} \cdot 2 \cdot (e^{2x}+1) - (e^{2x}-1) \cdot e^{2x} \cdot 2}{(e^{2x}+1)^2} \\ &= \frac{2e^{2x}(e^{2x}+1-e^{2x}+1)}{(e^{2x}+1)^2} \\ &= \frac{2e^{2x} \cdot 2}{(e^{2x}+1)^2} \\ &= \frac{4e^{2x}}{(e^{2x}+1)^2}. \end{aligned}$$

[5] (e) Because the function has the form $[f(x)]^{g(x)}$, we need to use logarithmic differentiation:

$$\begin{aligned}\ln(y) &= \ln(x^{\sqrt{x}}) \\ &= \sqrt{x}\ln(x).\end{aligned}$$

To differentiate the righthand side, we need the Product Rule:

$$\begin{aligned}\frac{d}{dx}[\ln(y)] &= \frac{d}{dx}[\sqrt{x}\ln(x)] \\ \frac{1}{y}\cdot\frac{dy}{dx} &= \frac{d}{dx}[x^{\frac{1}{2}}]\ln(x) + \sqrt{x}\cdot\frac{d}{dx}[\ln(x)] \\ &= \frac{1}{2}x^{-\frac{1}{2}}\cdot\ln(x) + \sqrt{x}\cdot\frac{1}{x} \\ &= \frac{\ln(x)}{2\sqrt{x}} + \frac{1}{\sqrt{x}} \\ &= \frac{\ln(x)+2}{2\sqrt{x}} \\ \frac{dy}{dx} &= y\cdot\frac{\ln(x)+2}{2\sqrt{x}} \\ &= x^{\sqrt{x}}\cdot\frac{\ln(x)+2}{2\sqrt{x}}.\end{aligned}$$

[5] (f) We use the Product Rule twice:

$$\begin{aligned}f'(x) &= [x^4]'\ln(x)\cot(x) + x^4[\ln(x)\cot(x)]' \\ &= 4x^3\ln(x)\cot(x) + x^4([\ln(x)]'\cot(x) + \ln(x)[\cot(x)]') \\ &= 4x^3\ln(x)\cot(x) + x^4\left(\frac{1}{x}\cdot\cot(x) + \ln(x)\cdot[-\csc^2(x)]\right) \\ &= 4x^3\ln(x)\cot(x) + x^3\cot(x) - x^4\ln(x)\csc^2(x).\end{aligned}$$

[5] 2. We use implicit differentiation, applying the Product Rule to the lefthand side:

$$\begin{aligned}
\frac{d}{dx}[x^3\cos(y)] &= \frac{d}{dx}[\sec(4x) - 6y] \\
\frac{d}{dx}[x^3]\cos(y) + x^3 \cdot \frac{d}{dx}[\cos(y)] &= \sec(4x)\tan(4x) \cdot \frac{d}{dx}[4x] - 6\frac{dy}{dx} \\
3x^2\cos(y) + x^3 \cdot \left[-\sin(y)\frac{dy}{dx}\right] &= \sec(4x)\tan(4x) \cdot 4 - 6\frac{dy}{dx} \\
6\frac{dy}{dx} - x^3\sin(y)\frac{dy}{dx} &= 4\sec(4x)\tan(4x) - 3x^2\cos(y) \\
\frac{dy}{dx}[6 - x^3\sin(y)] &= 4\sec(4x)\tan(4x) - 3x^2\cos(y) \\
\frac{dy}{dx} &= \frac{4\sec(4x)\tan(4x) - 3x^2\cos(y)}{6 - x^3\sin(y)}.
\end{aligned}$$

[5] 3. Let $A(x) = f(x) + g(x)$ so $A(x+h) = f(x+h) + g(x+h)$. Then

$$\begin{aligned}
[f(x) + g(x)]' = A'(x) &= \lim_{h\to 0} \frac{A(x+h) - A(x)}{h} \\
&= \lim_{h\to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h} \\
&= \lim_{h\to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} \\
&= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}\right] \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= f'(x) + g'(x).
\end{aligned}$$