MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

TEST 3 | **MATHEMATICS 1000-001** | FALL 2022

SOLUTIONS

[5] 1. (a) We use the Product Rule, followed by the Chain Rule:

$$\begin{aligned} y' &= [x^3]' \sin(\cot(x)) + x^3[\sin(\cot(x))]' \\ &= 3x^2 \sin(\cot(x)) + x^3 \cdot \cos(\cot(x)) \cdot [\cot(x)]' \\ &= 3x^2 \sin(\cot(x)) + x^3 \cos(\cot(x)) \cdot [-\csc^2(x)] \\ &= 3x^2 \sin(\cot(x)) - x^3 \cos(\cot(x)) \csc^2(x). \end{aligned}$$

[5] (b) We use the Chain Rule twice:

$$\begin{aligned} y' &= 3\sin^2(\cot(x)) \cdot [\sin(\cot(x))]' \\ &= 3\sin^2(\cot(x)) \cdot \cos(\cot(x)) \cdot [\cot(x)]' \\ &= 3\sin^2(\cot(x)) \cos(\cot(x)) \cdot [-\csc^2(x)] \\ &= -3\sin^2(\cot(x)) \cos(\cot(x)) \csc^2(x). \end{aligned}$$

[5] (c) We use the Chain Rule, followed by the Product Rule:

$$\begin{aligned} y' &= \cos(x^3 \cot(x)) \cdot [x^3 \cot(x)]' \\ &= \cos(x^3 \cot(x)) \cdot ([x^3]' \cot(x) + x^3[\cot(x)]') \\ &= \cos(x^3 \cot(x)) \cdot (3x^2 \cot(x) + x^3 \cdot [-\csc^2(x)]) \\ &= \cos(x^3 \cot(x)) \cdot (3x^2 \cot(x) - x^3 \csc^2(x)) \\ &= 3x^2 \cot(x) \cos(x^3 \cot(x)) - x^3 \csc^2(x) \cos(x^3 \cot(x)). \end{aligned}$$

[5] (d) We use the Product Rule twice:

$$\begin{aligned} f'(x) &= [x^2]' \ln(x) \tan(x) + x^2[\ln(x) \tan(x)]' \\ &= 2x \ln(x) \tan(x) + x^2([\ln(x)]' \tan(x) + \ln(x)[\tan(x)]') \\ &= 2x \ln(x) \tan(x) + x^2 \left(\frac{1}{x} \cdot \tan(x) + \ln(x) \cdot \sec^2(x)\right) \\ &= 2x \ln(x) \tan(x) + x \tan(x) + x^2 \ln(x) \sec^2(x). \end{aligned}$$

[5] (e) We use the Quotient Rule, followed by the Chain Rule:

$$\begin{aligned}
f'(x) &= \frac{[e^{5x}-1]'(e^{5x}+1)-(e^{5x}-1)[e^{5x}+1]'}{(e^{5x}+1)^2} \\
&= \frac{e^{5x}\cdot[5x]'\cdot(e^{5x}+1)-(e^{5x}-1)\cdot e^{5x}\cdot[5x]'}{(e^{5x}+1)^2} \\
&= \frac{e^{5x}\cdot 5\cdot(e^{5x}+1)-(e^{5x}-1)\cdot e^{5x}\cdot 5}{(e^{5x}+1)^2} \\
&= \frac{5e^{5x}(e^{5x}+1-e^{5x}+1)}{(e^{5x}+1)^2} \\
&= \frac{5e^{5x}\cdot 2}{(e^{5x}+1)^2} \\
&= \frac{10e^{5x}}{(e^{5x}+1)^2}.
\end{aligned}$$

[5] (f) Because the function has the form $[f(x)]^{g(x)}$, we need to use logarithmic differentiation:

$$\begin{aligned}
\ln(y) &= \ln(x^{\sqrt{x}}) \\
&= \sqrt{x}\ln(x).
\end{aligned}$$

To differentiate the righthand side, we need the Product Rule:

$$\begin{aligned}
\frac{d}{dx}[\ln(y)] &= \frac{d}{dx}[\sqrt{x}\ln(x)] \\
\frac{1}{y}\cdot\frac{dy}{dx} &= \frac{d}{dx}[x^{\frac{1}{2}}]\ln(x)+\sqrt{x}\cdot\frac{d}{dx}[\ln(x)] \\
&= \frac{1}{2}x^{-\frac{1}{2}}\cdot\ln(x)+\sqrt{x}\cdot\frac{1}{x} \\
&= \frac{\ln(x)}{2\sqrt{x}}+\frac{1}{\sqrt{x}} \\
&= \frac{\ln(x)+2}{2\sqrt{x}} \\
\frac{dy}{dx} &= y\cdot\frac{\ln(x)+2}{2\sqrt{x}} \\
&= x^{\sqrt{x}}\cdot\frac{\ln(x)+2}{2\sqrt{x}}.
\end{aligned}$$

[5] 2. We use implicit differentiation, applying the Product Rule to the lefthand side:

$$\begin{aligned}
\frac{d}{dx}[x^4\cos(y)] &= \frac{d}{dx}[\sec(2x) - 8y] \\
\frac{d}{dx}[x^4]\cos(y) + x^4 \cdot \frac{d}{dx}[\cos(y)] &= \sec(2x)\tan(2x) \cdot \frac{d}{dx}[2x] - 8\frac{dy}{dx} \\
4x^3\cos(y) + x^4 \cdot \left[-\sin(y)\frac{dy}{dx}\right] &= \sec(2x)\tan(2x) \cdot 2 - 8\frac{dy}{dx} \\
8\frac{dy}{dx} - x^4\sin(y)\frac{dy}{dx} &= 2\sec(2x)\tan(2x) - 4x^3\cos(y) \\
\frac{dy}{dx}[8 - x^4\sin(y)] &= 2\sec(2x)\tan(2x) - 4x^3\cos(y) \\
\frac{dy}{dx} &= \frac{2\sec(2x)\tan(2x) - 4x^3\cos(y)}{8 - x^4\sin(y)}.
\end{aligned}$$

[5] 3. Let $A(x) = f(x) - g(x)$ so $A(x+h) = f(x+h) - g(x+h)$. Then

$$\begin{aligned}
[f(x) - g(x)]' = A'(x) &= \lim_{h\to 0} \frac{A(x+h) - A(x)}{h} \\
&= \lim_{h\to 0} \frac{[f(x+h) - g(x+h)] - [f(x) - g(x)]}{h} \\
&= \lim_{h\to 0} \frac{f(x+h) - g(x+h) - f(x) + g(x)}{h} \\
&= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h} - \frac{g(x+h) - g(x)}{h}\right] \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} - \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= f'(x) - g'(x).
\end{aligned}$$