MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 3 | **MATHEMATICS 1000** | FALL 2024

SOLUTIONS

[6] 1. Since this is a quasirational function, we must consider both limits at infinity. Note that the smallest power of x in the denominator is effectively x (since we treat the x^2 inside the square root as having half its actual power). First, then,

$$\begin{aligned}
\lim_{x\to\infty} \frac{3-4x}{2x+\sqrt{16x^2-x-5}} &= \lim_{x\to\infty} \frac{3-4x}{2x+\sqrt{16x^2-x-5}} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \\
&= \lim_{x\to\infty} \frac{\frac{3}{x}-4}{2+\frac{1}{x}\sqrt{16x^2-x-5}} \\
&= \lim_{x\to\infty} \frac{\frac{3}{x}-4}{2+\frac{1}{\sqrt{x^2}}\sqrt{16x^2-x-5}} \\
&= \lim_{x\to\infty} \frac{\frac{3}{x}-4}{2+\sqrt{16-\frac{1}{x}-\frac{5}{x^2}}} \\
&= \frac{0-4}{2+\sqrt{16-0-0}} \\
&= \frac{-4}{6} \\
&= -\frac{2}{3}.
\end{aligned}$$

Next,

$$\begin{aligned}
\lim_{x\to-\infty} \frac{3-4x}{2x+\sqrt{16x^2-x-5}} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} &= \lim_{x\to-\infty} \frac{\frac{3}{x}-4}{2+\frac{1}{x}\sqrt{16x^2-x-5}} \\
&= \lim_{x\to-\infty} \frac{\frac{3}{x}-4}{2-\frac{1}{\sqrt{x^2}}\sqrt{16x^2-x-5}} \\
&= \lim_{x\to-\infty} \frac{\frac{3}{x}-4}{2-\sqrt{16-\frac{1}{x}-\frac{5}{x^2}}} \\
&= \frac{0-4}{2-\sqrt{16-0-0}} \\
&= \frac{-4}{-2} \\
&= 2.
\end{aligned}$$

Hence this function has two horizontal asymptotes: $y=-\frac{2}{3}$ and $y=2.$

[4] 2. First observe that $f(-2) = 9 - 2k$. This will be defined for all k.

Next,

$$\lim_{x\to-2} f(x) = \lim_{x\to-2}[(kx)^2 - 3kx + k] = \lim_{x\to-2}[k^2x^2 - 3kx + k] = 4k^2 + 6k + k = 4k^2 + 7k.$$

Thus the limit also exists for all k.

Finally, we need $\lim_{x\to-2} f(x) = f(-2)$, so we set

$$4k^2 + 7k = 9 - 2k$$
$$4k^2 + 9k - 9 = 0$$
$$(4k - 3)(k + 3) = 0$$

so $k = \frac{3}{4}$ or $k = -3$.

[2] 3. (a) We have $f(0) = -1$. The one-sided limits are

$$\lim_{x\to0^-} f(x) = \lim_{x\to0^-} \frac{x^2 + 5x + 4}{x^2 + 3x - 4} = -1 \quad \text{and} \quad \lim_{x\to0^+} f(x) = \lim_{x\to0^+}(x - 1) = -1.$$

Since the one-sided limits are equal, $\lim_{x\to0} f(x) = -1 = f(0)$, and hence the function is continuous at $x = 0$.

[2] (b) We have $f(2) = 1$. The one-sided limits are

$$\lim_{x\to2^-} f(x) = \lim_{x\to2^-}(x - 1) = 1 \quad \text{and} \quad \lim_{x\to2^+} f(x) = \lim_{x\to2^+} \frac{x^2 - 7x + 10}{x^2 - 10x + 25} = 0.$$

Since the one-sided limits disagree, $\lim_{x\to2} f(x)$ does not exist, and therefore $x = 2$ is a non-removable discontinuity.

[6] (c) Now we consider any values of x that would make any part of the definition of $f(x)$ undefined.

From the first definition, this occurs when

$$x^2 + 3x - 4 = (x + 4)(x - 1) = 0,$$

so $x = -4$ or $x = 1$. However, this definition only applies when $x \leq 0$, so we reject $x = 1$. When $x = -4$, direct substitution produces a $\frac{0}{0}$ indeterminate form, so we need to take the limit:

$$\lim_{x\to-4} f(x) = \lim_{x\to-4} \frac{x^2 + 5x + 4}{x^2 + 3x - 4} = \lim_{x\to-4} \frac{(x + 4)(x + 1)}{(x + 4)(x - 1)} = \lim_{x\to-4} \frac{x + 1}{x - 1} = \frac{3}{5}.$$

Since the limit exists, $x = -4$ is a removable discontinuity.

The second definition is a polynomial, which is always defined.

From the third definition, the denominator is zero when $x^2 - 10x + 25 = (x-5)^2 = 0$, so $x = 5$. Direct substitution results in a $\frac{0}{0}$ form, so again we must take the limit:

$$\lim_{x \to 5} f(x) = \lim_{x \to 5} \frac{x^2 - 7x + 10}{x^2 - 10x + 25} = \lim_{x \to 5} \frac{(x-5)(x-2)}{(x-5)^2} = \lim_{x \to 5} \frac{x-2}{x-5},$$

which results in a $\frac{3}{0}$ form. Thus the limit does not exist, and $x = 5$ is a non-removable discontinuity.