MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 3 **MATHEMATICS 1000** FALL 2022

SOLUTIONS

[6] 1. In the denominator, the largest power of x outside the square root is x, while inside the square root it is x^2, which treat as $x^{\frac{1}{2}\cdot 2} = x$ as well. Either way, the dominant term is x. Since this is a quasirational function, we need to consider both limits at infinity. First, as $x \to \infty$, recall that $x = \sqrt{x^2}$ and so

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{7-x}{5x+\sqrt{16x^2-3x+4}} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \\
&= \lim_{x\to\infty} \frac{\frac{7}{x}-1}{5+\frac{1}{x}\cdot\sqrt{16x^2-3x+4}} \\
&= \lim_{x\to\infty} \frac{\frac{7}{x}-1}{5+\frac{1}{\sqrt{x^2}}\cdot\sqrt{16x^2-3x+4}} \\
&= \lim_{x\to\infty} \frac{\frac{7}{x}-1}{5+\sqrt{16-\frac{3}{x}+\frac{4}{x^2}}} \\
&= \frac{0-1}{5+\sqrt{16-0+0}} \\
&= -\frac{1}{9}.
\end{aligned}$$

As $x \to -\infty$, $x = -\sqrt{x^2}$ so

$$\begin{aligned}
\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} \frac{7-x}{5x+\sqrt{16x^2-3x+4}} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \\
&= \lim_{x\to-\infty} \frac{\frac{7}{x}-1}{5+\frac{1}{x}\cdot\sqrt{16x^2-3x+4}} \\
&= \lim_{x\to-\infty} \frac{\frac{7}{x}-1}{5+\frac{1}{-\sqrt{x^2}}\cdot\sqrt{16x^2-3x+4}} \\
&= \lim_{x\to-\infty} \frac{\frac{7}{x}-1}{5-\sqrt{16-\frac{3}{x}+\frac{4}{x^2}}} \\
&= \frac{0-1}{5-\sqrt{16-0+0}} \\
&= -1.
\end{aligned}$$

Hence $f(x)$ has two horizontal asymptotes: $y = -\frac{1}{9}$ and $y = -1$.

[6] 2. First observe that

$$f(3) = \frac{3k}{3k+5},$$

and this is defined for all $k \neq -\frac{5}{3}$.

Next, we have

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (kx^2 - 2kx) = 9k - 6k = 3k$$

while

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (4x - 9k^2) = 12 - 9k^2.$$

In order for $f(x)$ to be continuous at $x = 3$, the one-sided limits must be equal, so we set

$$3k = 12 - 9k^2$$
$$9k^2 + 3k - 12 = 0$$
$$3(3k+4)(k-1) = 0,$$

and therefore either $k = -\frac{4}{3}$ or $k = 1$.

When $k = -\frac{4}{3}$,

$$f(3) = -4 \quad \text{and} \quad \lim_{x\to 3} f(x) = -4,$$

so $f(x)$ is continuous at $x = 3$.

When $k = 1$,

$$f(3) = \frac{3}{8} \quad \text{and} \quad \lim_{x\to 3} f(x) = 3,$$

so $f(x)$ is not continuous at $x = 3$.

Hence $k = -\frac{4}{3}$ is the only solution.

[2] 3. (a) We have

$$f(-2) = (-2)^2 + 3(-2) - 6 = -8.$$

The one-sided limits are

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} \frac{x^2+6x+9}{x^2-9} = -\frac{1}{5} \quad \text{and} \quad \lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} (x^2+3x-6) = -8.$$

Thus $\lim_{x\to -2} f(x)$ does not exist, and so $x = -2$ is a non-removable discontinuity.

[2] (b) We have

$$f(1) = \frac{1^2 + 1 - 20}{1^2 - 8(1) + 16} = -2$$

The one-sided limits are

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^2+3x-6) = -2 \quad \text{and} \quad \lim_{x\to 1^-} \frac{x^2+x-20}{x^2-8x+16} = -2,$$

so

$$\lim_{x\to 1} f(x) = -2 = f(1),$$

and thus $f(x)$ is in fact continuous at $x = 1$.

[4] (c) Now we consider any values of x that would make any part of the definition of $f(x)$ undefined.

The first part will be undefined if

$$x^2 - 9 = 0 \quad \implies \quad (x-3)(x+3) = 0$$

that is, if $x = 3$ or $x = -3$. However, because this part of the definition applies only for $x < -2$, we reject $x = 3$. Thus $x = -3$ is a discontinuity, and because

$$\lim_{x \to -3} f(x) = \lim_{x \to -3} \frac{x^2 + 6x + 9}{x^2 - 9} = \lim_{x \to -3} \frac{(x+3)^2}{(x-3)(x+3)} = \lim_{x \to -3} \frac{x+3}{x-3} = 0,$$

the discontinuity at $x = -3$ is removable.

The second part of the definition is a polynomial, and is therefore never undefined.

The third part is undefined if

$$x^2 - 8x + 16 = 0 \quad \implies \quad (x-4)^2 = 0,$$

that is, if $x = 4$. Since this part of the definition applies for $x \geq 1$, then, $x = 4$ is a discontinuity. Because

$$\lim_{x \to 4} f(x) = \lim_{x \to 4} \frac{x^2 + x - 20}{x^2 - 8x + 16} = \lim_{x \to 4} \frac{(x-4)(x+5)}{(x-4)^2} = \lim_{x \to 4} \frac{x+5}{x-4}$$

does not exist (since a $\frac{K}{0}$ form results), the discontinuity at $x = 4$ is non-removable.