

MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

TEST 2

MATHEMATICS 1000-006

FALL 2024

SOLUTIONS

[8] 1. (a) By the limit definition of the derivative,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{(x+h)^2}{3(x+h)+1} - \frac{x^2}{3x+1}}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \cdot \left(\frac{x^2 + 2xh + h^2}{3x + 3h + 1} - \frac{x^2}{3x + 1} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \cdot \frac{(x^2 + 2xh + h^2)(3x + 1) - x^2(3x + 3h + 1)}{(3x + 1)(3x + 3h + 1)} \\ &= \lim_{h \rightarrow 0} \frac{3x^3 + x^2 + 6x^2h + 2xh + 3xh^2 + h^2 - 3x^3 - 3x^2h - x^2}{h(3x + 1)(3x + 3h + 1)} \\ &= \lim_{h \rightarrow 0} \frac{3x^2h + 2xh + 3xh^2 + h^2}{h(3x + 1)(3x + 3h + 1)} \\ &= \lim_{h \rightarrow 0} \frac{3x^2 + 2x + 3xh + h}{(3x + 1)(3x + 3h + 1)} \\ &= \frac{3x^2 + 2x}{(3x + 1)^2}. \end{aligned}$$

[3] (b) By the Quotient Rule,

$$\begin{aligned} f'(x) &= \frac{[x^2]'(3x + 1) - x^2[3x + 1]'}{(3x + 1)^2} \\ &= \frac{2x \cdot (3x + 1) - x^2 \cdot 3}{(3x + 1)^2} \\ &= \frac{6x^2 + 2x - 3x}{(3x + 1)^2} \\ &= \frac{3x^2 + 2x}{(3x + 1)^2}. \end{aligned}$$

[5] 2. (a) We use the Product Rule:

$$y' = [x^{\frac{1}{2}}]' \sec(\sin(x)) + \sqrt{x}[\sec(\sin(x))]'.$$

For the derivative in the second term, we need the Chain Rule:

$$\begin{aligned} y' &= \frac{1}{2}x^{-\frac{1}{2}} \sec(\sin(x)) + \sqrt{x} \sec(\sin(x)) \tan(\sin(x)) [\sin(x)]' \\ &= \frac{\sec(\sin(x))}{2\sqrt{x}} + \sqrt{x} \sec(\sin(x)) \tan(\sin(x)) \cos(x). \end{aligned}$$

[5] (b) We use the Chain Rule twice:

$$\begin{aligned} y' &= 5 \cot^4(3x) [\cot(3x)]' \\ &= 5 \cot^4(x) \cdot [-\csc^2(3x)] [3x]' \\ &= 5 \cot^4(x) \cdot [-\csc^2(3x)] \cdot 3 \\ &= -15 \cot^4(3x) \csc^2(3x). \end{aligned}$$

[5] (c) We use the Chain Rule, followed by the Product Rule:

$$\begin{aligned} y' &= -\sin(x^7 e^x) [x^7 e^x]' \\ &= -\sin(x^7 e^x) \cdot ([x^7]' e^x + x^7 [e^x]') \\ &= -\sin(x^7 e^x) \cdot (7x^6 e^x + x^7 e^x) \\ &= -7x^6 e^x \sin(x^7 e^x) - x^7 e^x \sin(x^7 e^x). \end{aligned}$$

[7] 3. (a) We differentiate both sides of the equation with respect to x , using the Product Rule on the lefthand side:

$$\begin{aligned} \frac{d}{dx}[xy^4 + 9] &= \frac{d}{dx}[x^2 + 3y] \\ \frac{d}{dx}[x]y^4 + x \cdot \frac{d}{dx}[y^4] + 0 &= 2x + 3\frac{dy}{dx} \\ 1 \cdot y^4 + x \cdot 4y^3 \frac{dy}{dx} &= 2x + 3\frac{dy}{dx} \\ 4xy^3 \frac{dy}{dx} - 3\frac{dy}{dx} &= 2x - y^4 \\ \frac{dy}{dx}(4xy^3 - 3) &= 2x - y^4 \\ \frac{dy}{dx} &= \frac{2x - y^4}{4xy^3 - 3}. \end{aligned}$$

[3] (b) The slope of the tangent line is

$$m = \frac{2 \cdot 3 - 1^4}{4 \cdot 3 \cdot 1^3 - 3} = \frac{5}{9}.$$

Hence the equation of the tangent line is

$$y - 1 = \frac{5}{9}(x - 3)$$

$$y - 1 = \frac{5}{9}x - \frac{5}{3}$$

$$y = \frac{5}{9}x - \frac{2}{3}.$$

[5] 4. Let $A(x) = kf(x)$. Then

$$\begin{aligned}[kf(x)]' &= A'(x) = \lim_{h \rightarrow 0} \frac{A(x+h) - A(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{kf(x+h) - kf(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{k[f(x+h) - f(x)]}{h} \\ &= \lim_{h \rightarrow 0} k \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= kf'(x).\end{aligned}$$