TEST 2 | **MATHEMATICS 1000-001** | FALL 2024

SOLUTIONS

[8] 1. (a) By the limit definition of the derivative,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{(x+h)^2}{3(x+h)+1} - \frac{x^2}{3x+1}}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\cdot\left(\frac{x^2+2xh+h^2}{3x+3h+1} - \frac{x^2}{3x+1}\right) \\ &= \lim_{h\to 0} \frac{1}{h}\cdot\frac{(x^2+2xh+h^2)(3x+1) - x^2(3x+3h+1)}{(3x+1)(3x+3h+1)} \\ &= \lim_{h\to 0} \frac{3x^3+x^2+6x^2h+2xh+3xh^2+h^2-3x^3-3x^2h-x^2}{h(3x+1)(3x+3h+1)} \\ &= \lim_{h\to 0} \frac{3x^2h+2xh+3xh^2+h^2}{h(3x+1)(3x+3h+1)} \\ &= \lim_{h\to 0} \frac{3x^2+2x+3xh+h}{(3x+1)(3x+3h+1)} \\ &= \frac{3x^2+2x}{(3x+1)^2}. \end{aligned}$$

[3] (b) By the Quotient Rule,

$$\begin{aligned} f'(x) &= \frac{[x^2]'(3x+1) - x^2[3x+1]'}{(3x+1)^2} \\ &= \frac{2x\cdot(3x+1) - x^2\cdot 3}{(3x+1)^2} \\ &= \frac{6x^2+2x-3x}{(3x+1)^2} \\ &= \frac{3x^2+2x}{(3x+1)^2}. \end{aligned}$$

[5] 2. (a) We use the Chain Rule, followed by the Product Rule:

$$\begin{aligned} y' &= -\sin(x^7e^x)[x^7e^x]' \\ &= -\sin(x^7e^x)\cdot([x^7]'e^x + x^7[e^x]') \\ &= -\sin(x^7e^x)\cdot(7x^6e^x + x^7e^x) \\ &= -7x^6e^x\sin(x^7e^x) - x^7e^x\sin(x^7e^x). \end{aligned}$$

[5] (b) We use the Product Rule:

$$y' = [x^{\frac{1}{2}}]' \sec(\sin(x)) + \sqrt{x}[\sec(\sin(x))]'.$$

For the derivative in the second term, we need the Chain Rule:

$$y' = \frac{1}{2}x^{-\frac{1}{2}} \sec(\sin(x)) + \sqrt{x}\sec(\sin(x))\tan(\sin(x))[\sin(x)]'$$

$$= \frac{\sec(\sin(x))}{2\sqrt{x}} + \sqrt{x}\sec(\sin(x))\tan(\sin(x))\cos(x).$$

[5] (c) We use the Chain Rule twice:

$$\begin{aligned} y' &= 5\cot^4(3x)[\cot(3x)]' \\ &= 5\cot^4(x) \cdot [-\csc^2(3x)][3x]' \\ &= 5\cot^4(x) \cdot [-\csc^2(3x)] \cdot 3 \\ &= -15\cot^4(3x)\csc^2(3x). \end{aligned}$$

[7] 3. (a) We differentiate both sides of the equation with respect to x, using the Product Rule on the lefthand side:

$$\begin{aligned} \frac{d}{dx}[xy^4 + 9] &= \frac{d}{dx}[x^2 + 3y] \\ \frac{d}{dx}[x]y^4 + x \cdot \frac{d}{dx}[y^4] + 0 &= 2x + 3\frac{dy}{dx} \\ 1 \cdot y^4 + x \cdot 4y^3\frac{dy}{dx} &= 2x + 3\frac{dy}{dx} \\ 4xy^3\frac{dy}{dx} - 3\frac{dy}{dx} &= 2x - y^4 \\ \frac{dy}{dx}(4xy^3 - 3) &= 2x - y^4 \\ \frac{dy}{dx} &= \frac{2x - y^4}{4xy^3 - 3}. \end{aligned}$$

[3] (b) The slope of the tangent line is

$$m = \frac{2 \cdot 3 - 1^4}{4 \cdot 3 \cdot 1^3 - 3} = \frac{5}{9}.$$

Hence the equation of the tangent line is

$$\begin{aligned} y - 1 &= \frac{5}{9}(x - 3) \\ y - 1 &= \frac{5}{9}x - \frac{5}{3} \\ y &= \frac{5}{9}x - \frac{2}{3}. \end{aligned}$$

[5] 4. Let $A(x) = f(x) - g(x)$. Then

$$\begin{aligned}
[f(x) - g(x)]' = A'(x) &= \lim_{h\to 0} \frac{A(x+h) - A(x)}{h} \\
&= \lim_{h\to 0} \frac{[f(x+h) - g(x+h)] - [f(x) - g(x)]}{h} \\
&= \lim_{h\to 0} \frac{f(x+h) - g(x+h) - f(x) + g(x)}{h} \\
&= \lim_{h\to 0} \left(\frac{f(x+h) - f(x)}{h} - \frac{g(x+h) - g(x)}{h} \right) \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} - \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= f'(x) - g'(x).
\end{aligned}$$