MEMORIAL UNIVERSITY OF NEWFOUNDLAND DEPARTMENT OF MATHEMATICS AND STATISTICS

Assignment 2

MATHEMATICS 1000

Fall 2024

SOLUTIONS

[4] 1. (a) This is a quasirational function for which direct substitution yields a $\frac{0}{0}$ indeterminate form, so we use the Rationalisation Method. There is a radical in *both* the numerator and the denominator, so let's first try rationalising the numerator:

$$\lim_{x \to -7} \frac{5 - \sqrt{4 - 3x}}{\sqrt{x + 8} - 1} \cdot \frac{5 + \sqrt{4 - 3x}}{5 + \sqrt{4 - 3x}} = \lim_{x \to -7} \frac{25 - (4 - 3x)}{(\sqrt{x + 8} - 1)(5 + \sqrt{4 - 3x})}$$
$$= \lim_{x \to -7} \frac{3x + 21}{(\sqrt{x + 8} - 1)(5 + \sqrt{4 - 3x})}.$$

Now we'll rationalise the denominator:

$$\lim_{x \to -7} \frac{3x + 21}{(\sqrt{x + 8} - 1)(5 + \sqrt{4} - 3x)} \cdot \frac{\sqrt{x + 8} + 1}{\sqrt{x + 8} + 1}$$

$$= \lim_{x \to -7} \frac{(3x + 21)(\sqrt{x + 8} + 1)}{[(x + 8) - 1](5 + \sqrt{4} - 3x)}$$

$$= \lim_{x \to -7} \frac{3(x + 7)(\sqrt{x + 8} + 1)}{(x + 7)(5 + \sqrt{4} - 3x)}$$

$$= \lim_{x \to -7} \frac{3(\sqrt{x + 8} + 1)}{5 + \sqrt{4} - 3x}$$

$$= \frac{6}{10}$$

$$= \frac{3}{5}.$$

[3] (b) Direct substitution yields another type of indeterminate form $(\infty - \infty)$ so we first need to rewrite the given function in a way that will allow us to use the techniques we've

learned. We have

$$\lim_{h \to -2} [(h+5)(h+2)^{-1} + 21(h^2 - 3h - 10)^{-1}]$$

$$= \lim_{h \to -2} \left(\frac{h+5}{h+2} + \frac{21}{h^2 - 3h - 10}\right)$$

$$= \lim_{h \to -2} \left(\frac{h+5}{h+2} + \frac{21}{(h-5)(h+2)}\right)$$

$$= \lim_{h \to -2} \left(\frac{(h-5)(h+5)}{(h-5)(h+2)} + \frac{21}{(h-5)(h+2)}\right)$$

$$= \lim_{h \to -2} \frac{(h-5)(h+5) + 21}{(h-5)(h+2)}$$

$$= \lim_{h \to -2} \frac{h^2 - 4}{(h-5)(h+2)}$$

Now we've obtained a rational function (and note that direct substitution produces a $\frac{0}{0}$ indeterminate form) so we can use the Cancellation Method:

$$\frac{h^2 - 4}{(h - 5)(h + 2)} = \lim_{h \to -2} \frac{(h - 2)(h + 2)}{(h - 5)(h + 2)}$$
$$= \lim_{h \to -2} \frac{h - 2}{h - 5}$$
$$= \frac{-4}{-7}$$
$$= \frac{4}{7}.$$

[4] (c) Again, direct substitution results in a $\frac{0}{0}$ indeterminate form. But recall that, for any θ ,

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}.$$

This means that we can rewrite the given limit as

$$\lim_{x \to 0} \frac{\tan^4(3x)}{x^4} = \lim_{x \to 0} \frac{\left(\frac{\sin(3x)}{\cos(3x)}\right)^4}{x^4}$$
$$= \lim_{x \to 0} \frac{1}{\cos^4(3x)} \cdot \lim_{x \to 0} \frac{\sin^4(3x)}{x^4}$$
$$= \frac{1}{\cos^4(0)} \cdot \lim_{x \to 0} \frac{\sin^4(3x)}{x^4}$$
$$= 1 \cdot \lim_{x \to 0} \frac{\sin^4(3x)}{x^4}$$
$$= \lim_{x \to 0} \frac{\sin^4(3x)}{x^4}.$$

Now we can try using the special trigonometric limit

$$\lim_{t \to 0} \frac{\sin(t)}{t} = 1.$$

Observe that

$$\lim_{x \to 0} \frac{\sin(3x)}{x} = \lim_{x \to 0} \frac{3\sin(3x)}{3x} = 3\lim_{x \to 0} \frac{\sin(3x)}{3x}$$

As $x \to 0$, $3x \to 0$ so this limit has the same form as the special sine limit with t = 3x. Thus

$$\lim_{x \to 0} \frac{\sin(3x)}{x} = 3 \cdot 1 = 3.$$

But then

$$\lim_{x \to 0} \frac{\sin^4(3x)}{x^4} = \left(\lim_{x \to 0} \frac{\sin(3x)}{x}\right)^4 = 3^4 = 81.$$

[3] 2. Since this is a piecewise function whose behaviour changes at x = 3, we must check the one-sided limits:

$$\lim_{x \to 3^{-}} f(x) = \lim_{x \to 3^{-}} (x^2 - k^2) = 9 - k^2$$

and

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (kx + 5) = 3k + 5.$$

If the limit exists, then these one-sided limits must be equal, so we set

$$9 - k^{2} = 3k + 5$$
$$k^{2} + 3k - 4 = 0$$
$$(k + 4)(k - 1) = 0,$$

and hence k = -4 or k = 1.

(Note that the third part of the definition of f(x) did not affect our workings because it applies only for x < 0, which is far away from x = 3.)

[6] 3. To find the vertical asymptotes, we first set the denominator equal to zero and solve for x:

$$6 - x - x^{2} = 0$$
$$x^{2} + x - 6 = 0$$
$$x - 2)(x + 3) = 0,$$

so the only possible vertical asymptotes are at x = 2 and x = -3.

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At x = 2, the numerator is $3(2^2) + 13(2) + 12 = 50$. Since this is non-zero, it guarantees that x = 2 is a vertical asymptote.

At x = -3, the numerator is $3(-3)^2 + 13(-3) + 12 = 0$. Thus we have a $\frac{0}{0}$ indeterminate form at x = -3. This means that we must take the limit to see if a vertical asymptote is located there:

$$\lim_{x \to -3} \frac{3x^2 + 13x + 12}{6 - x - x^2} = \lim_{x \to -3} \frac{(3x + 4)(x + 3)}{-(x - 2)(x + 3)} = \lim_{x \to -3} \frac{3x + 4}{2 - x} = -1$$

Since the limit exists, x = -3 is not a vertical asymptote (it must just be a hole in the graph).

Finally, observe that

$$\lim_{x \to 2^{-}} \frac{3x^2 + 13x + 12}{6 - x - x^2} = \lim_{x \to 2^{-}} \frac{3x + 4}{2 - x} = \infty$$

since the numerator is positive and the denominator is small and positive as $x \to 2^-$. Similarly,

$$\lim_{x \to 2^+} \frac{3x^2 + 13x + 12}{6 - x - x^2} = \lim_{x \to 2^+} \frac{3x + 4}{2 - x} = -\infty$$

since the numerator is positive and the denominator is small and negative as $x \to 2^+$.