MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Test 1 | **MATHEMATICS 1000-006** | Fall 2024

SOLUTIONS

[12] 1. (a) $f(-3)$ $= -4$

(b) $\lim_{x \to -3^-} f(x)$ $= 0$

(c) $\lim_{x \to -3^+} f(x)$ $= 0$

(d) $\lim_{x \to -3} f(x)$ $= 0$

(e) $f(2)$ is undefined

(f) $\lim_{x \to 2^-} f(x)$ $= -\infty$

(g) $\lim_{x \to 2^+} f(x)$ $= -\infty$

(h) $\lim_{x \to 2} f(x)$ $= -\infty$ (and therefore does not exist)

(i) $f(0)$ $= 2$

(j) $\lim_{x \to 0^-} f(x)$ $= -2$

(k) $\lim_{x \to 0^+} f(x)$ $= 2$

(l) $\lim_{x \to 0} f(x)$ does not exist (because the one-sided limits are not equal)

[4] 2. (a) We obtain a $\frac{0}{0}$ indeterminate form by direct substitution. But recall the special limit

$$\lim_{t \to 0} \frac{\sin(t)}{t} = 1.$$

We can rewrite the given limit as

$$\begin{aligned}
\lim_{x \to 0} \frac{\sin^2(5x)}{x^2} &= \lim_{x \to 0} \left[\frac{\sin(5x)}{x}\right]^2 \\
&= \left[\lim_{x \to 0} \frac{\sin(5x)}{x}\right]^2 \\
&= \left[\lim_{x \to 0} \frac{\sin(5x)}{x} \cdot \frac{5}{5}\right]^2 \\
&= 25\left[\lim_{x \to 0} \frac{\sin(5x)}{5x}\right]^2 \\
&= 25 \cdot 1^2 \\
&= 25.
\end{aligned}$$

[6] (b) Direct substitution yields a difference of $\frac{K}{0}$ but, since the resulting limits would not exist, we cannot rewrite this as the difference of two limits. Instead, we rewrite the given function as a single rational function, and then use the Cancellation Method:

$$\begin{aligned}
\lim_{x\to-3}\left[\frac{6}{x^2-9}-\frac{1}{x^2+5x+6}\right] &= \lim_{x\to-3}\left[\frac{6}{(x-3)(x+3)}-\frac{1}{(x+2)(x+3)}\right] \\
&= \lim_{x\to-3}\frac{6(x+2)-(x-3)}{(x-3)(x+2)(x+3)} \\
&= \lim_{x\to-3}\frac{5x+15}{(x-3)(x+2)(x+3)} \\
&= \lim_{x\to-3}\frac{5(x+3)}{(x-3)(x+2)(x+3)} \\
&= \lim_{x\to-3}\frac{5}{(x-3)(x+2)} \\
&= \frac{5}{-6\cdot(-1)} \\
&= \frac{5}{6}.
\end{aligned}$$

[6] (c) Direct substitution results in a $\frac{0}{0}$ indeterminate form. Since this is a quasirational function, we use the Rationalisation Method:

$$\begin{aligned}
\lim_{x\to2}\frac{3x-6}{3-\sqrt{5x-1}}\cdot\frac{3+\sqrt{5x-1}}{3+\sqrt{5x-1}} &= \lim_{x\to2}\frac{(3x-6)\left(3+\sqrt{5x-1}\right)}{9-(5x-1)} \\
&= \lim_{x\to2}\frac{3(x-2)\left(3+\sqrt{5x-1}\right)}{10-5x} \\
&= \lim_{x\to2}\frac{3(x-2)\left(3+\sqrt{5x-1}\right)}{-5(x-2)} \\
&= \lim_{x\to2}\frac{3\left(3+\sqrt{5x-1}\right)}{-5} \\
&= \frac{3(3+3)}{-5} \\
&= -\frac{18}{5}.
\end{aligned}$$

[4] 3. Because $f(x)$ is a rational function, we need evaluate only one of the limits at infinity:

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{6x^2+7}{x(1-3x)} \\
&= \lim_{x\to\infty} \frac{6x^2+7}{x-3x^2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \\
&= \lim_{x\to\infty} \frac{6+\frac{7}{x^2}}{\frac{1}{x}-3} \\
&= \lim_{x\to\infty} \frac{6+0}{0-3} \\
&= -2.
\end{aligned}$$

Hence the only horizontal asymptote is the line $\boxed{y=-2}$.

[8] 4. (a) We have

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (kx+7) = 3k+7$$

and

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (x+k^2) = 3+k^2.$$

In order for the limit to exist, we require

$$\begin{aligned}
\lim_{x\to 3^-} f(x) &= \lim_{x\to 3^+} f(x) \\
3k+7 &= 3+k^2 \\
k^2-3k-4 &= 0 \\
(k-4)(k+1) &= 0
\end{aligned}$$

so $\boxed{k=4}$ or $\boxed{k=-1}$.

For $k=4$, observe that

$$f(3) = 2\sqrt{4+5} = 6 \quad \text{and} \quad \lim_{x\to 3} f(x) = 19.$$

For $k=-1$, we see that

$$f(3) = 2\sqrt{-1+5} = 4 \quad \text{and} \quad \lim_{x\to 3} f(x) = 4.$$

(b) Since $f(3) = \lim_{x\to 3} f(x)$ when $\boxed{k=-1}$ we may conclude that $f(x)$ is continuous at $x=3$ for this value of k.

(c) Since $f(3) \neq \lim_{x\to 3} f(x)$ but $\lim_{x\to 3} f(x)$ exists when $\boxed{k=4}$ we may conclude that $f(x)$ has a removable discontinuity at $x=3$ for this value of k.