MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

TEST 1 | **MATHEMATICS 1000-003** | FALL 2022

SOLUTIONS

[12] 1. (a) $f(2)$ is undefined

(b) $\lim\limits_{x \to 2^-} f(x) = \infty$

(c) $\lim\limits_{x \to 2^+} f(x) = \infty$

(d) $\lim\limits_{x \to 2} f(x) = \infty$ (and therefore the limit does not exist)

(e) $f(0) = 2$

(f) $\lim\limits_{x \to 0^-} f(x) = 2$

(g) $\lim\limits_{x \to 0^+} f(x) = -3$

(h) $\lim\limits_{x \to 0} f(x)$ does not exist (because the one-sided limits do not agree)

(i) $f(-2) = -4$

(j) $\lim\limits_{x \to -2^-} f(x) = 0$

(k) $\lim\limits_{x \to -2^+} f(x) = 0$

(l) $\lim\limits_{x \to -2} f(x) = 0$

[6] 2. (a) Since $f(x)$ changes its definition at $x = -3$, we must consider the one-sided limits. From the left,

$$\lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} \frac{6x + 18}{9 - x^2}.$$

Direct substitution produces a $\frac{0}{0}$ indeterminate form, so we apply the Cancellation Method:

$$\begin{aligned}
\lim_{x \to -3^-} f(x) &= \lim_{x \to -3^-} \frac{6(x+3)}{-(x-3)(x+3)} \\
&= \lim_{x \to -3^-} \frac{6}{-(x-3)} \\
&= \frac{6}{-(-6)} \\
&= 1.
\end{aligned}$$

From the right, direct substitution immediately provides

$$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} \frac{3 - 5x}{9 + x^2} = \frac{3 - (-15)}{9 + 9} = 1.$$

Since the one-sided limits are equal, we can conclude that

$$\lim_{x \to -3} f(x) = 1.$$

[5] (b) Direct substitution yields a $\frac{0}{0}$ indeterminate form, so we use the Rationalisation Method:

$$\begin{aligned}
\lim_{x\to 2}\frac{x^2-2x}{3x-\sqrt{7x^2+8}} &= \lim_{x\to 2}\frac{x^2-2x}{3x-\sqrt{7x^2+8}}\cdot\frac{3x+\sqrt{7x^2+8}}{3x+\sqrt{7x^2+8}}\\
&= \lim_{x\to 2}\frac{(x^2-2x)(3x+\sqrt{7x^2+8})}{9x^2-7x^2-8}\\
&= \lim_{x\to 2}\frac{(x^2-2x)(3x+\sqrt{7x^2+8})}{2x^2-8}\\
&= \lim_{x\to 2}\frac{x(x-2)(3x+\sqrt{7x^2+8})}{2(x-2)(x+2)}\\
&= \lim_{x\to 2}\frac{x(3x+\sqrt{7x^2+8})}{2(x+2)}\\
&= \frac{2(6+\sqrt{36})}{2\cdot 4}\\
&= 3.
\end{aligned}$$

[5] (c) Direct substitution yields a $\frac{0}{0}$ indeterminate form, but we can rewrite the given limit as

$$\begin{aligned}
\lim_{x\to 0}\frac{\sin^2(3x)}{x\sin(4x)} &= \lim_{x\to 0}\frac{\sin(3x)}{x}\cdot\lim_{x\to 0}\frac{\sin(3x)}{\sin(4x)}\\
&= \lim_{x\to 0}\frac{\sin(3x)}{x}\cdot\frac{3}{3}\cdot\lim_{x\to 0}\frac{\sin(3x)}{\sin(4x)}\\
&= 3\lim_{x\to 0}\frac{\sin(3x)}{3x}\cdot\lim_{x\to 0}\frac{\sin(3x)}{\sin(4x)}\\
&= 3\cdot 1\cdot\lim_{x\to 0}\frac{\sin(3x)}{\sin(4x)}\cdot\frac{3x}{3x}\\
&= 3\cdot 3\lim_{x\to 0}\frac{\sin(3x)}{3x}\cdot\lim_{x\to 0}\frac{x}{\sin(4x)}\\
&= 9\cdot 1\cdot\lim_{x\to 0}\frac{x}{\sin(4x)}\cdot\frac{4}{4}\\
&= 9\cdot\frac{1}{4}\lim_{x\to 0}\frac{4x}{\sin(4x)}\\
&= \frac{9}{4}\cdot 1\\
&= \frac{9}{4}.
\end{aligned}$$

[8] 3. We set

$$2x^2 - 5x + 3 = 0$$
$$(2x-3)(x-1) = 0$$

so the possible vertical asymptotes are $x = \frac{3}{2}$ and $x = 1$.

First we observe that $f(\frac{3}{2}) = \frac{0}{0}$ so we must find out whether the limit exists in order to determine if $x = \frac{3}{2}$ is a vertical asymptote. By the Cancellation Method,

$$\begin{aligned}
\lim_{x\to\frac{3}{2}} \frac{3-2x}{2x^2-5x+3} &= \lim_{x\to\frac{3}{2}} \frac{-(2x-3)}{(2x-3)(x-1)} \\
&= \lim_{x\to\frac{3}{2}} \frac{-1}{x-1} \\
&= \frac{-1}{\frac{1}{2}} \\
&= -2.
\end{aligned}$$

Since the limit exists, $x = \frac{3}{2}$ is not a vertical asymptote.

Next, $f(1) = \frac{1}{0}$, so $x = 1$ is a vertical asymptote. Furthermore, we can observe that $x - 1$ is a small negative number when $x \to 1^-$ so

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{-1}{x-1} = \infty.$$

Likewise, $x - 1$ is a small positive number when $x \to 1^+$ so

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} \frac{-1}{x-1} = -\infty.$$

[4] 4. (a) This statement is false. We can write

$$f(x) = \frac{x^2-9}{x-3} = \frac{(x-3)(x+3)}{x-3} = x+3 = g(x)$$

only when $x - 3 \neq 0$, that is, when $x \neq 3$. In fact, $f(3)$ is undefined (because it results in division by zero) while $g(3) = 6$. Thus these functions are not equal.

(b) This statement is true. Because $f(x)$ and $g(x)$ differ only at $x = 3$, and the limit as $x \to 3$ is not affected by the behaviour of the functions at $x = 3$, their limits must be equal. Indeed,

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3}(x+3) = 6$$

by the Cancellation Method, while

$$\lim_{x\to 3} g(x) = \lim_{x\to 3}(x+3) = 6$$

by direct substitution.