

## SOLUTIONS

- [12] 1. (a)  $f(0) = 2$   
(b)  $\lim_{x \rightarrow 0^-} f(x) = 2$   
(c)  $\lim_{x \rightarrow 0^+} f(x) = -3$   
(d)  $\lim_{x \rightarrow 0} f(x)$  does not exist (because the one-sided limits do not agree)  
(e)  $f(-2) = -4$   
(f)  $\lim_{x \rightarrow -2^-} f(x) = 0$   
(g)  $\lim_{x \rightarrow -2^+} f(x) = 0$   
(h)  $\lim_{x \rightarrow -2} f(x) = 0$   
(i)  $f(2)$  is undefined  
(j)  $\lim_{x \rightarrow 2^-} f(x) = \infty$   
(k)  $\lim_{x \rightarrow 2^+} f(x) = \infty$   
(l)  $\lim_{x \rightarrow 2} f(x) = \infty$  (and therefore the limit does not exist)

- [5] 2. (a) Direct substitution yields a  $\frac{0}{0}$  indeterminate form, so we use the Rationalisation Method:

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{x^2 - 2x}{3x - \sqrt{7x^2 + 8}} &= \lim_{x \rightarrow 2} \frac{x^2 - 2x}{3x - \sqrt{7x^2 + 8}} \cdot \frac{3x + \sqrt{7x^2 + 8}}{3x + \sqrt{7x^2 + 8}} \\ &= \lim_{x \rightarrow 2} \frac{(x^2 - 2x)(3x + \sqrt{7x^2 + 8})}{9x^2 - 7x^2 - 8} \\ &= \lim_{x \rightarrow 2} \frac{(x^2 - 2x)(3x + \sqrt{7x^2 + 8})}{2x^2 - 8} \\ &= \lim_{x \rightarrow 2} \frac{x(x - 2)(3x + \sqrt{7x^2 + 8})}{2(x - 2)(x + 2)} \\ &= \lim_{x \rightarrow 2} \frac{x(3x + \sqrt{7x^2 + 8})}{2(x + 2)} \\ &= \frac{2(6 + \sqrt{36})}{2 \cdot 4} \\ &= 3. \end{aligned}$$

[5] (b) Direct substitution yields a  $\frac{0}{0}$  indeterminate form, but we can rewrite the given limit as

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin^2(3x)}{x \sin(4x)} &= \lim_{x \rightarrow 0} \frac{\sin(3x)}{x} \cdot \lim_{x \rightarrow 0} \frac{\sin(3x)}{\sin(4x)} \\ &= \lim_{x \rightarrow 0} \frac{\sin(3x)}{x} \cdot \frac{3}{3} \cdot \lim_{x \rightarrow 0} \frac{\sin(3x)}{\sin(4x)} \\ &= 3 \lim_{x \rightarrow 0} \frac{\sin(3x)}{3x} \cdot \lim_{x \rightarrow 0} \frac{\sin(3x)}{\sin(4x)} \\ &= 3 \cdot 1 \cdot \lim_{x \rightarrow 0} \frac{\sin(3x)}{\sin(4x)} \cdot \frac{3x}{3x} \\ &= 3 \cdot 3 \lim_{x \rightarrow 0} \frac{\sin(3x)}{3x} \cdot \lim_{x \rightarrow 0} \frac{x}{\sin(4x)} \\ &= 9 \cdot 1 \cdot \lim_{x \rightarrow 0} \frac{x}{\sin(4x)} \cdot \frac{4}{4} \\ &= 9 \cdot \frac{1}{4} \lim_{x \rightarrow 0} \frac{4x}{\sin(4x)} \\ &= \frac{9}{4} \cdot 1\end{aligned}$$

$$= \frac{9}{4}.$$

[6] (c) Since  $f(x)$  changes its definition at  $x = -3$ , we must consider the one-sided limits. From the left,

$$\lim_{x \rightarrow -3^-} f(x) = \lim_{x \rightarrow -3^-} \frac{6x + 18}{9 - x^2}.$$

Direct substitution produces a  $\frac{0}{0}$  indeterminate form, so we apply the Cancellation Method:

$$\begin{aligned}\lim_{x \rightarrow -3^-} f(x) &= \lim_{x \rightarrow -3^-} \frac{6(x + 3)}{-(x - 3)(x + 3)} \\ &= \lim_{x \rightarrow -3^-} \frac{6}{-(x - 3)} \\ &= \frac{6}{-(-6)} \\ &= 1.\end{aligned}$$

From the right, direct substitution immediately provides

$$\lim_{x \rightarrow -3^+} f(x) = \lim_{x \rightarrow -3^+} \frac{3 - 5x}{9 + x^2} = \frac{3 - (-15)}{9 + 9} = 1.$$

Since the one-sided limits are equal, we can conclude that

$$\lim_{x \rightarrow -3} f(x) = 1.$$

[8] 3. We set

$$\begin{aligned}2x^2 - 5x + 3 &= 0 \\(2x - 3)(x - 1) &= 0\end{aligned}$$

so the possible vertical asymptotes are  $x = \frac{3}{2}$  and  $x = 1$ .

First we observe that  $f(\frac{3}{2}) = \frac{0}{0}$  so we must find out whether the limit exists in order to determine if  $x = \frac{3}{2}$  is a vertical asymptote. By the Cancellation Method,

$$\begin{aligned}\lim_{x \rightarrow \frac{3}{2}} \frac{3 - 2x}{2x^2 - 5x + 3} &= \lim_{x \rightarrow \frac{3}{2}} \frac{-(2x - 3)}{(2x - 3)(x - 1)} \\&= \lim_{x \rightarrow \frac{3}{2}} \frac{-1}{x - 1} \\&= \frac{-1}{\frac{1}{2}} \\&= -2.\end{aligned}$$

Since the limit exists,  $x = \frac{3}{2}$  is not a vertical asymptote.

Next,  $f(1) = \frac{1}{0}$ , so  $x = 1$  is a vertical asymptote. Furthermore, we can observe that  $x - 1$  is a small negative number when  $x \rightarrow 1^-$  so

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} \frac{-1}{x - 1} = \infty.$$

Likewise,  $x - 1$  is a small positive number when  $x \rightarrow 1^+$  so

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \frac{-1}{x - 1} = -\infty.$$

[4] 4. (a) This statement is **false**. We can write

$$f(x) = \frac{x^2 - 9}{x - 3} = \frac{(x - 3)(x + 3)}{x - 3} = x + 3 = g(x)$$

only when  $x - 3 \neq 0$ , that is, when  $x \neq 3$ . In fact,  $f(3)$  is undefined (because it results in division by zero) while  $g(3) = 6$ . Thus these functions are not equal.

(b) This statement is **true**. Because  $f(x)$  and  $g(x)$  differ only at  $x = 3$ , and the limit as  $x \rightarrow 3$  is not affected by the behaviour of the functions at  $x = 3$ , their limits must be equal. Indeed,

$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = \lim_{x \rightarrow 3} \frac{(x - 3)(x + 3)}{x - 3} = \lim_{x \rightarrow 3} (x + 3) = 6$$

by the Cancellation Method, while

$$\lim_{x \rightarrow 3} g(x) = \lim_{x \rightarrow 3} (x + 3) = 6$$

by direct substitution.