Section 4.6: L'Hôpital's Rule

Theorem: L'Hôpital's Rule

Suppose we wish to evaluate $\lim_{x \to p} \frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are differentiable functions and $g'(x) \neq 0$ near $x = p$ (except possibly at $x = p$). Further suppose that either $\lim_{x \to p} f(x) = \lim_{x \to p} g(x) = 0$, or that $\lim_{x \to p} f(x)$ and $\lim_{x \to p} g(x)$ are both infinite. Then

$$\lim_{x \to p} \frac{f(x)}{g(x)} = \lim_{x \to p} \frac{f'(x)}{g'(x)}$$

if the limit on the right exists or is infinite.

eg $\lim_{x \to 3} \frac{x-3}{x^2-9}$ ($\frac{0}{0}$ form)

We could use the Cancellation Method:

$$\lim_{x \to 3} \frac{x-3}{(x-3)(x+3)} = \lim_{x \to 3} \frac{1}{x+3} = \frac{1}{6}$$

But now we can instead apply l'Hôpital's Rule:

$$\lim_{x \to 3} \frac{x-3}{x^2-9} \overset{H}{=} \lim_{x \to 3} \frac{[x-3]'}{[x^2-9]'}$$

$$= \lim_{x \to 3} \frac{1}{2x} \boxed{= \frac{1}{6}}$$

L'Hôpital's Rule: $\lim\limits_{x \to p} \frac{f(x)}{g(x)} = \lim\limits_{x \to p} \frac{f'(x)}{g'(x)}$

eg $\lim\limits_{x \to 0} \frac{\sin(x)}{x}$ ($\frac{0}{0}$ form)

$\overset{H}{=} \lim\limits_{x \to 0} \frac{[\sin(x)]'}{[x]'}$

$= \lim\limits_{x \to 0} \frac{\cos(x)}{1} = \cos(0) \boxed{= 1}$

We can also apply l'Hôpital's Rule to limits at infinity.

eg $\lim\limits_{x \to \infty} \frac{4x+1}{3-x}$ ($\frac{\infty}{\infty}$ form)

By our methods from Unit 1, we would write

$\lim\limits_{x \to \infty} \frac{4x+1}{3-x} \cdot \frac{1/x}{1/x} = \lim\limits_{x \to \infty} \frac{4 + 1/x}{3/x - 1}$

$= \frac{4+0}{0-1} = -4$

By l'Hôpital's Rule, we have

$\lim\limits_{x \to \infty} \frac{4x+1}{3-x} \overset{H}{=} \lim\limits_{x \to \infty} \frac{4}{-1} \boxed{= -4}$

eg $\lim\limits_{x \to \infty} \frac{1-x}{e^{3x}}$ ($\frac{\infty}{\infty}$ form)

$\overset{H}{=} \lim\limits_{x \to \infty} \frac{-1}{3e^{3x}} \boxed{= 0}$

We can also apply l'Hôpital's Rule to one-sided limits.

eg $$\lim_{x \to 1^-} \frac{x^2-1}{\arcsin(x) - \frac{\pi}{2}} \quad \left(\tfrac{0}{0} \text{ form}\right)$$

$$\overset{H}{=} \lim_{x \to 1^-} \frac{2x}{1/\sqrt{1-x^2}}$$

$$= \lim_{x \to 1^-} 2x\sqrt{1-x^2} \quad \boxed{= 0}$$

We often need to apply l'Hôpital's Rule more than once to evaluate a given limit.

eg $$\lim_{x \to 0} \frac{1-\cos^2(x)}{x^2} \quad \left(\tfrac{0}{0} \text{ form}\right)$$

$$\overset{H}{=} \lim_{x \to 0} \frac{-2\cos(x)\cdot[-\sin(x)]}{2x}$$

$$= \lim_{x \to 0} \frac{\cos(x)\sin(x)}{x} \quad \left(\tfrac{0}{0} \text{ form}\right)$$

$$\overset{H}{=} \lim_{x \to 0} \frac{-\sin^2(x) + \cos^2(x)}{1} \quad \boxed{= 1}$$

We can also use l'Hôpital's Rule to show that a limit is infinite.

eg $$\lim_{x \to 0^+} \frac{x}{e^{x^2}-1} \quad \left(\tfrac{0}{0} \text{ form}\right)$$

$$\overset{H}{=} \lim_{x \to 0^+} \frac{1}{e^{x^2}\cdot 2x} \quad \boxed{= \infty}$$

L'Hôpital's Rule is sometimes much less efficient than our earlier methods.

eg $$\lim_{x\to\infty} \frac{6x^4+7x^3+4x}{3x^4+2x^2+9} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$\overset{H}{=} \lim_{x\to\infty} \frac{24x^3+21x^2+4}{12x^3+4x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$\overset{H}{=} \lim_{x\to\infty} \frac{72x^2+42x}{36x^2+4} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$\overset{H}{=} \lim_{x\to\infty} \frac{144x+42}{72x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$\overset{H}{=} \lim_{x\to\infty} \frac{144}{72} \boxed{=2}$$

It is much easier to write

$$\lim_{x\to\infty} \frac{6x^4+7x^3+4x}{3x^4+2x^2+9} \cdot \frac{1/x^4}{1/x^4}$$

$$= \lim_{x\to\infty} \frac{6+7/x+4/x^3}{3+2/x^2+9/x^4} = \frac{6+0+0}{3+0+0} = 2$$

If we apply l'Hôpital's Rule to a limit where we do <u>not</u> have a $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form then we can obtain an erroneous result.

eg $\lim_{x\to 0} \frac{\cos(x)}{\sin(x)+1} \overset{H}{=} \lim_{x\to 0} \frac{-\sin(x)}{\cos(x)}$

$= 0$

However, direct substitution into the original limit gives

$$\lim_{x\to 0} \frac{\cos(x)}{\sin(x)+1} = \frac{1}{0+1} \boxed{= 1}$$

But how could we apply l'Hôpital's Rule to the other indeterminate forms: $0\cdot\infty$, $\infty-\infty$, 0^0, ∞^0, 1^∞?

We must first rewrite the limit as a $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form.

eg $\lim_{x\to 0^+} x\ln(x)$ ($0\cdot\infty$ form)

First we rewrite the function as

$$\lim_{x\to 0^+} \frac{\ln(x)}{1/x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$\overset{H}{=} \lim_{x\to 0^+} \frac{1/x}{-1/x^2}$$

$$= \lim_{x\to 0^+} (-x) \boxed{= 0}$$

eg $\lim_{x\to \frac{\pi}{2}^-} (\pi-2x)\tan(x)$ ($0\cdot\infty$ form)

$$= \lim_{x\to \frac{\pi}{2}^-} \frac{\pi-2x}{\cot(x)} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$\overset{H}{=} \lim_{x\to\frac{\pi}{2}^-} \frac{-2}{-\csc^2(x)}$$

$$= \lim_{x\to\frac{\pi}{2}^-} 2\sin^2(x) \boxed{= 2}$$

eg $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sinh(x)}\right)$ ($\infty - \infty$ form)

$$= \lim_{x\to 0} \left(\frac{\sinh(x)}{x\sinh(x)} - \frac{x}{x\sinh(x)}\right)$$

$$= \lim_{x\to 0} \frac{\sinh(x) - x}{x\sinh(x)} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$\overset{H}{=} \lim_{x\to 0} \frac{\cosh(x) - 1}{\sinh(x) + x\cosh(x)} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$\overset{H}{=} \lim_{x\to 0} \frac{\sinh(x)}{\cosh(x) + \cosh(x) + x\sinh(x)}$$

$$= \lim_{x\to 0} \frac{\sinh(x)}{2\cosh(x) + x\sinh(x)} = \frac{0}{2} \boxed{= 0}$$

Given a limit where direct substitution results in an exponential indeterminate form, we first consider the limit of the logarithm of the given function and try to apply l'Hôpital's Rule.

eg $\lim_{x \to \infty} x^{1/x}$ (∞^0 form)

Let $y = x^{1/x}$

$$\ln(y) = \ln(x^{1/x}) = \frac{1}{x}\ln(x) = \frac{\ln(x)}{x}$$

Then $\lim_{x \to \infty} \ln(y) = \lim_{x \to \infty} \frac{\ln(x)}{x}$ ($\frac{\infty}{\infty}$ form)

$$\overset{H}{=} \lim_{x \to \infty} \frac{1/x}{1} = 0$$

But now we can write

$$\lim_{x \to \infty} x^{1/x} = \lim_{x \to \infty} y$$

$$= \lim_{x \to \infty} e^{\ln(y)}$$

$$= e^{\lim_{x \to \infty} \ln(y)}$$

$$= e^{0}$$

$$\boxed{= 1}$$

eg $\lim_{x \to 0^+} (2x+1)^{\csc(x)}$ (1^∞ form)

Let $y = (2x+1)^{\csc(x)}$

$$\ln(y) = \ln\left((2x+1)^{\csc(x)}\right)$$

$$= \csc(x) \ln(2x+1)$$

$$= \frac{\ln(2x+1)}{\sin(x)}$$

Then $\lim_{x \to 0^+} \ln(y) = \lim_{x \to 0^+} \frac{\ln(2x+1)}{\sin(x)}$ ($\frac{0}{0}$ form)

$$\overset{H}{=} \lim_{x \to 0^+} \frac{\frac{1}{2x+1} \cdot 2}{\cos(x)}$$

$$= \lim_{x \to 0^+} \frac{2}{(2x+1)\cos(x)} = \frac{2}{1} = 2$$

Hence $\lim_{x \to 0^+} y = \lim_{x \to 0^+} (2x+1)^{\csc(x)} \boxed{= e^2}$.

eg $\lim_{x \to 0^+} (5x)^{\tan(x)}$ (0^0 form)

Let $y = (5x)^{\tan(x)}$

$$\ln(y) = \ln\left((5x)^{\tan(x)}\right) = \tan(x) \ln(5x)$$

$$= \frac{\ln(5x)}{\cot(x)}$$

We have $\lim_{x\to 0^+} \ln(y) = \lim_{x\to 0^+} \frac{\ln(5x)}{\cot(x)}$ $\left(\frac{\infty}{\infty} \text{ form}\right)$

$$\overset{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{5x}\cdot 5}{-\csc^2(x)}$$

$$= \lim_{x\to 0^+} \frac{-\sin^2(x)}{x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$\overset{H}{=} \lim_{x\to 0^+} \frac{-2\sin(x)\cos(x)}{1}$$

$$= 0$$

Finally, $\lim_{x\to 0^+} y = \lim_{x\to 0^+} (5x)^{\tan(x)} = e^0 \boxed{=1}$.

eg Sketch the graph of $f(x) = \frac{2\ln(x)}{x^2}$.

The domain is $x > 0$, and $x = 0$ is a vertical asymptote because $\lim_{x\to 0^+} \frac{2\ln(x)}{x^2} = -\infty$.

Next, we have

$$\lim_{x\to\infty} \frac{2\ln(x)}{x^2} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$\overset{H}{=} \lim_{x\to\infty} \frac{2/x}{2x}$$

$$= \lim_{x\to\infty} \frac{1}{x^2}$$

$$= 0$$

Thus $y = 0$ is the only horizontal asymptote.

To find any x-intercepts, we set

$$f(x) = 0$$

$$2\ln(x) = 0$$

$$\ln(x) = 0 \rightarrow x = e^0 = 1$$

so $(1,0)$ is the only x-intercept.

There are no y-intercepts because $x=0$ is not in the domain of $f(x)$.

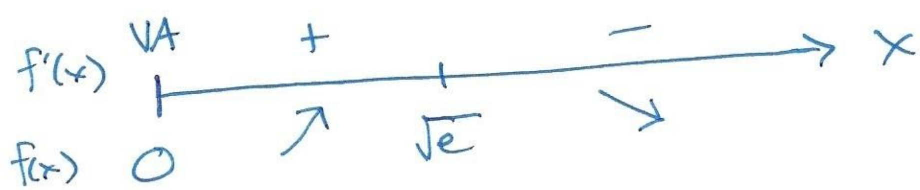
Next, $f'(x) = \dfrac{2-4\ln(x)}{x^3}$ and since $f'(x)$ is never undefined on the domain of $f(x)$, we set

$$f'(x) = 0$$

$$2 - 4\ln(x) = 0$$

$$\ln(x) = \tfrac{1}{2} \rightarrow x = e^{1/2} = \sqrt{e} \quad \text{(CRITICAL POINT)}$$

$$\approx 1.6$$



Thus $f(x)$ is increasing for $0 < x < \sqrt{e}$
decreasing for $x > \sqrt{e}$

Also, $x = \sqrt{e}$ is a relative maximum; here,

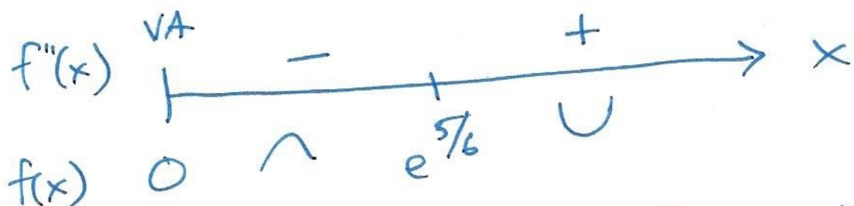
$$y = \frac{2\ln(\sqrt{e})}{(\sqrt{e})^2} = \frac{1}{e} \approx 0.4$$

Finally, $f''(x) = \dfrac{12\ln(x) - 10}{x^4}$ and since $f''(x)$ is never undefined on the domain of $f(x)$, we set

$$f''(x) = 0$$

$$12\ln(x) - 10 = 0$$

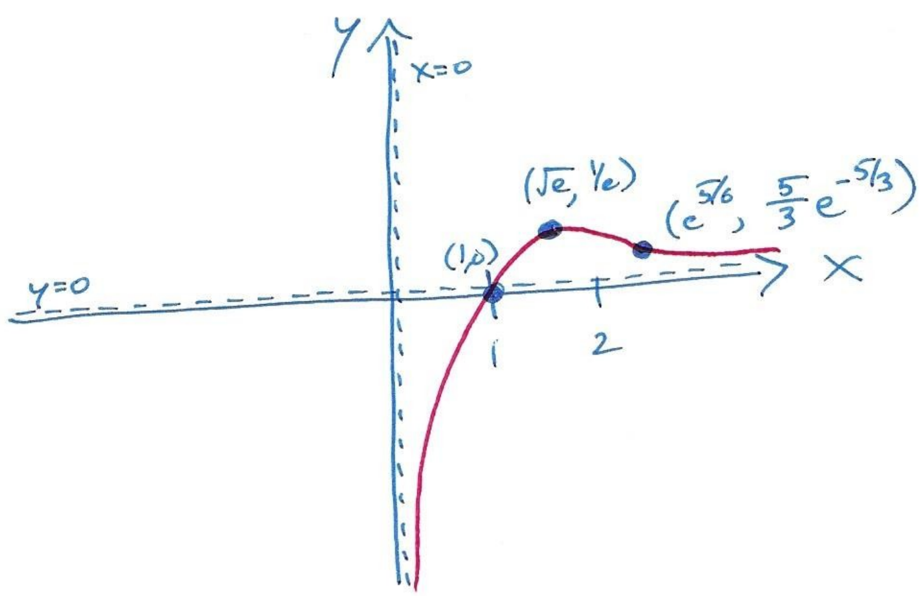
$$\ln(x) = \frac{5}{6} \rightarrow x = e^{5/6} \approx 2.3 \quad \left(\begin{array}{c}\text{HYPERCRITICAL} \\ \text{POINT}\end{array}\right)$$



Thus $f(x)$ is concave upward for $x > e^{5/6}$
concave downward for $0 < x < e^{5/6}$

Also, $x = e^{5/6}$ is a point of inflection; here,

$$y = \frac{2\ln(e^{5/6})}{(e^{5/6})^2} = \frac{5/3}{e^{5/3}} \approx 0.3 .$$



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