Section 4.5: Optimisation Problems

We often want to maximize or minimize a quantity, usually subject to certain constraints. This is called an <u>optimization</u> or <u>max-min</u> problem.

eg The sum of two non-negative numbers is 6. What is the largest possible value of the sum of their squares?

Let the two numbers be x and y. We are given that $x \geq 0$ and $y \geq 0$, and $x+y=6$. We want to maximise $S = x^2+y^2$.

Unfortunately, $S = x^2 + y^2$ expresses S as a function of both x and y.

However, we know that $x + y = 6$ so $y = 6 - x$.

Thus

$$\begin{aligned} S(x) &= x^2 + (6-x)^2 \\ &= x^2 + 36 - 12x + x^2 \\ &= 2x^2 - 12x + 36 \end{aligned}$$

We know that $x \geq 0$ and, since $y \geq 0$ as well, this means that $6 - x \geq 0$ so $x \leq 6$. Thus $S(x)$ is defined on the <u>closed</u> interval $0 \leq x \leq 6$. Hence we use the Extreme Value Theorem.

$$S'(x) = 4x - 12$$

We set

$$S'(x) = 0$$

$$4x - 12 = 0 \rightarrow x = 3 \quad \text{(CRITICAL POINT)}$$

$$S(3) = 18$$

$$S(0) = 36 \qquad S(6) = 36$$

The largest possible value of the sum of the squares is <u>36</u>, which happens when the two numbers are 0 and 6.

Strategy for optimisation problems

① Identify all variables needed, all given information, and the quantity to be maximised or minimised.

② Establish the primary equation, which expresses the quantity to be maximised or minimised in terms of the other variables.

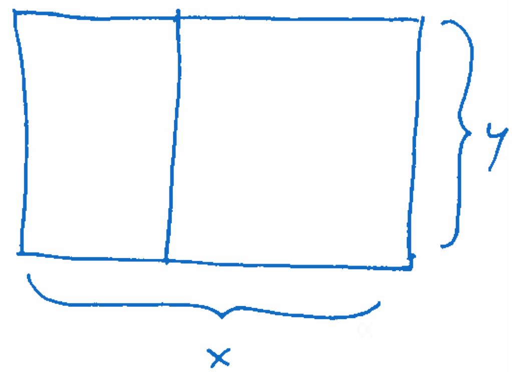
③ Use the constraints of the problem to identify one or more secondary equations to reduce the primary equation to an expression involving the quantity to be maximised or minimised and exactly one other variable.

④ If the resulting function is continuous and defined on a closed interval, we apply the Extreme Value Theorem.

If it is continuous, defined on an open (or half-open) interval and has a single critical point on that interval, we apply the Second Derivative Test.

eg A farmer with 120 feet of fencing wants to enclose a rectangular area and then divide it into two pens with fencing parallel to one side of the rectangle. What is the largest possible total area of the two pens?

Let L be the total length of fencing.



Let x be the length of one side of the fence and y be the length of the other (parallel to the internal divider).

Let A be the total area (which is to be maximised).

We are given that $L = 120$.

We know that $A = xy$.

We also have
$$\begin{aligned} L = 2x + 3y &= 120 \\ 3y &= 120 - 2x \\ y &= 40 - \tfrac{2}{3}x \end{aligned}$$

Hence
$$\begin{aligned} A(x) &= x\left(40 - \tfrac{2}{3}x\right) \\ &= 40x - \tfrac{2}{3}x^2 \qquad \text{where } x > 0 \end{aligned}$$

We hope to use the Second Derivative Test.

We have $A'(x) = 40 - \frac{4}{3}x$ so we set

$$\begin{aligned} A'(x) &= 0 \\ 40 - \tfrac{4}{3}x &= 0 \\ \tfrac{4}{3}x &= 40 \\ x &= 30 \end{aligned}$$ (only critical point)

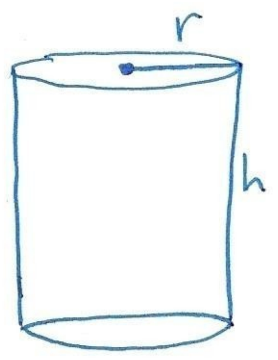
Next, $A''(x) = -\frac{4}{3} < 0$ for all x including $x = 30$.

By the Second Derivative Test, $x = 30$ is the absolute maximum.

Finally, $A(30) = 1200 - 600 = 600$

Thus the maximum area of the two pens is $\underline{600 \text{ ft}^2}$.

eg A closed cylindrical can is to hold 2π litres of liquid. How should we choose the height and the radius of the can to minimize the amount of material needed for its manufacture?



Let V, r, h be the volume, radius, height of the cylinder. Let M be the amount of material.

We are given that $V = 2\pi \cdot 1000 = 2000\pi$.

We want to minimize M.

The primary equation is $M = 2\pi r^2 + 2\pi rh$

The secondary equation is $V = \pi r^2 h = 2000\pi$

$$h = \frac{2000}{r^2}$$

The reduced primary equation is

$$M(r) = 2\pi r^2 + 2\pi r \cdot \frac{2000}{r^2}$$

$$= 2\pi r^2 + \frac{4000\pi}{r} \quad \text{where } r > 0$$

$$M'(r) = 4\pi r - \frac{4000\pi}{r^2}$$

We set $M'(r) = 0$

$$4\pi r - \frac{4000\pi}{r^2} = 0$$

$$4\pi r = \frac{4000\pi}{r^2}$$

$$r^3 = 1000 \longrightarrow r = 10 \quad \text{(CRITICAL POINT)}$$

Next, $M''(r) = 4\pi + \dfrac{8000\pi}{r^3}$

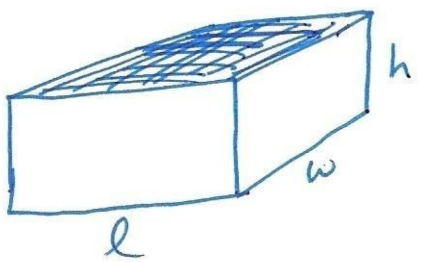
$$M''(10) = 4\pi + \frac{8000\pi}{10^3} = 4\pi + 8\pi = 12\pi > 0$$

Thus $r = 10$ is the absolute minimum by the Second Derivative Test.

From the secondary equation, $h = \dfrac{2000}{10^2} = 20$.

Hence the amount of material is minimized when the radius is <u>10 cm</u> and the height is <u>20 cm</u>.

eg An open-topped box is to have a volume of $20 \ m^3$. All the sides of the box are rectangular, and its length is to be twice its width. Material for the base of the box costs \$12 per square metre, while material for the sides costs \$4 per square metre. Find the cost of the materials for the cheapest such box.



Let V, l, w, h be the volume, length, width, height of the box. Let C be the cost of the materials.

We are given that $V = 20$. We want to minimise C.

The primary equation is

$$C = \underbrace{12lw}_{\text{BASE}} + \underbrace{2(4lh)}_{\text{FRONT / BACK SIDES}} + \underbrace{2(4wh)}_{\text{RIGHT / LEFT SIDES}}$$

$$= 12lw + 8lh + 8wh$$

We need two secondary equations:

$$V = lwh = 20$$

$$l = 2w \longrightarrow V = (2w)wh = 20$$

$$h = \frac{10}{w^2}$$

We can now reduce the primary equation to

$$C(w) = 12(2w)w + 8(2w)\cdot\frac{10}{w^2} + 8w\cdot\frac{10}{w^2}$$

$$= 24w^2 + \frac{160}{w} + \frac{80}{w}$$

$$= 24w^2 + \frac{240}{w} \quad \text{where } w > 0$$

$$C'(w) = 48w - \frac{240}{w^2}$$

We set $C'(w) = 0$

$$48w - \frac{240}{w^2} = 0$$

$$w^3 = 5 \longrightarrow w = \sqrt[3]{5} \quad \text{(CRITICAL POINT)}$$

$$C''(w) = 48 + \frac{480}{w^3}$$

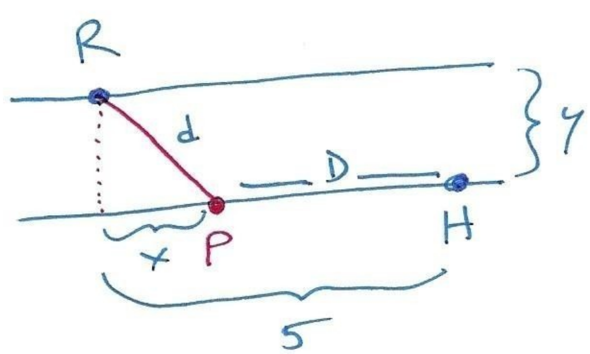
$$C''(\sqrt[3]{5}) = 48 + \frac{480}{5} > 0$$

By the Second Derivative Test, $w = \sqrt[3]{5}$ is the absolute minimum.

Then $C(\sqrt[3]{5}) = 24(\sqrt[3]{5})^2 + \frac{240}{\sqrt[3]{5}} \approx 227.33$.

The minimum cost is $\underline{\$227.33}$.

eg Marshal Wyatt Earp is closing in on Johnny Ringo. Ringo wants to reach a hiding place on the opposite side of the San Pedro River, 5 miles downstream. All along this stretch, the river is 1 mile wide. Ringo can swim at 2 miles per hour but run at 3 miles per hour. To one decimal place, where should Ringo come ashore in order to reach the hiding place as quickly as possible?



Let y be the width of the river, let d be the distance Ringo swims, and D be the distance Ringo runs. Let T be his total travel time.

We are given that $y = 1$. We want to minimize T.

The primary equation is

$$T = T_{SWIM} + T_{RUN}$$

$$= \frac{d}{2} + \frac{D}{3}$$

Now let x be the distance at which Ringo comes ashore. By the Pythagorean theorem,

$$x^2 + y^2 = d^2$$

$$x^2 + 1 = d^2 \rightarrow d = \sqrt{x^2+1}$$

But also, $x + D = 5 \rightarrow D = 5 - x$.

Thus the primary equation becomes

$$T = \frac{\sqrt{x^2+1}}{2} + \frac{5-x}{3}$$

$$T(x) = \frac{1}{2}\sqrt{x^2+1} + \frac{5}{3} - \frac{1}{3}x \quad \text{where } 0 \le x \le 5.$$

$$T'(x) = \frac{1}{2}\frac{1}{2}(x^2+1)^{-1/2} \cdot 2x + 0 - \frac{1}{3}$$

$$= \frac{x}{2\sqrt{x^2+1}} - \frac{1}{3}$$

Now we set $T'(x) = 0$

$$\frac{x}{2\sqrt{x^2+1}} - \frac{1}{3} = 0$$

$$3x = 2\sqrt{x^2+1}$$

$$9x^2 = 4(x^2+1)$$

$$5x^2 = 4 \longrightarrow x^2 = \frac{4}{5} \rightarrow x = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

(CRITICAL POINT)

$$T\left(\frac{2\sqrt{5}}{5}\right) \approx 2.04$$

$$T(0) \approx 2.17$$

$$T(5) \approx 2.55$$

Hence $x = \frac{2\sqrt{5}}{5}$ is the absolute minimum. Thus Ringo should come ashore $\frac{2\sqrt{5}}{5} \approx$ <u>0.9 miles</u> downstream.