Section 4.4: Absolute Extrema

Def'n: A function $f(x)$ defined on an interval I has an absolute (or global) maximum at $x=p$ if $f(p) \geq f(x)$ for all x in I. It has an absolute (or global) minimum at $x=p$ if $f(p) \leq f(x)$ for all x in I.

The value of $f(x)$ at an absolute maximum is the maximum value. Its value at an absolute minimum is the minimum value. These are both unique.

The Extreme Value Theorem

If a function $f(x)$ is continuous on a closed interval $a \leq x \leq b$ then $f(x)$ possesses both a maximum value and a minimum value on that interval.

Given $f(x)$ which is continuous on $a \le x \le b$, we identify the maximum and minimum values as follows:

1. find all the critical points of $f(x)$ on the interval $a < x < b$
2. evaluate $f(x)$ at each critical point
3. evaluate $f(a)$ and $f(b)$

The greatest of these values is the maximum value, and the least is the minimum value.

eg Find the maximum and minimum values of

$$f(x) = (x+2)^2(x-3)^3$$

on $-3 \le x \le 1$.

In Section 4.2, we found the critical points to be $x=0$, $x=-2$, $x=3$. However, we omit $x=3$ because it does not lie on the indicated closed interval.

$f(0) = -108$ $\quad$ $f(-2) = 0$

$f(-3) = -216$ $\quad$ $f(1) = -72$

Thus the maximum value is $\underline{0}$, and the minimum value is $\underline{-216}$.

eg Find the maximum and minimum values of

$$f(x) = \frac{8x}{x^2+4}$$

on $0 \le x \le 6$.

We have $f'(x) = \dfrac{-8(x-2)(x+2)}{(x^2+4)^2}$ which is never undefined, so we set

OMIT

$$f'(x) = 0$$
$$-8(x-2)(x+2) = 0 \to x = 2, \quad x = -2 \quad \text{(CRITICAL POINTS)}$$

Next, $f(2) = \frac{16}{8} = 2$ $\qquad f(6) = \frac{48}{40} = \frac{6}{5}$

$f(0) = 0$

Thus $\underline{2}$ is the maximum value and $\underline{0}$ is the minimum value.

Theorem: The Second Derivative Test

Let $x=p$ be the only critical point of a function $f(x)$ on an open interval I. If $f(x)$, $f'(x)$ and $f''(x)$ are all continuous on I then

1. if $f''(p) < 0$ then $x=p$ is the absolute maximum of $f(x)$ on I

2. if $f''(p) > 0$ then $x=p$ is the absolute minimum of $f(x)$ on I

eg Determine the maximum value of

$$f(x) = x + 2\cos(x)$$

on $0 < x < \frac{\pi}{2}$.

We have $f'(x) = 1 - 2\sin(x)$ so we set

$$\begin{aligned} f'(x) &= 0 \\ 1 - 2\sin(x) &= 0 \\ \sin(x) &= \tfrac{1}{2} \\ x &= \arcsin\left(\tfrac{1}{2}\right) = \frac{\pi}{6} \end{aligned}$$

Because there is only one critical point, the SDT applies. Thus

$$f''(x) = -2\cos(x)$$
$$f''\left(\tfrac{\pi}{6}\right) = -2\cos\left(\tfrac{\pi}{6}\right) = -2 \cdot \frac{\sqrt{3}}{2} = -\sqrt{3} < 0$$

Hence $x = \frac{\pi}{6}$ is the absolute maximum, and so the maximum value of $f(x)$ is

$$f\left(\tfrac{\pi}{6}\right) = \frac{\pi}{6} + 2\cos\left(\tfrac{\pi}{6}\right)$$
$$\boxed{= \frac{\pi}{6} + \sqrt{3}}$$

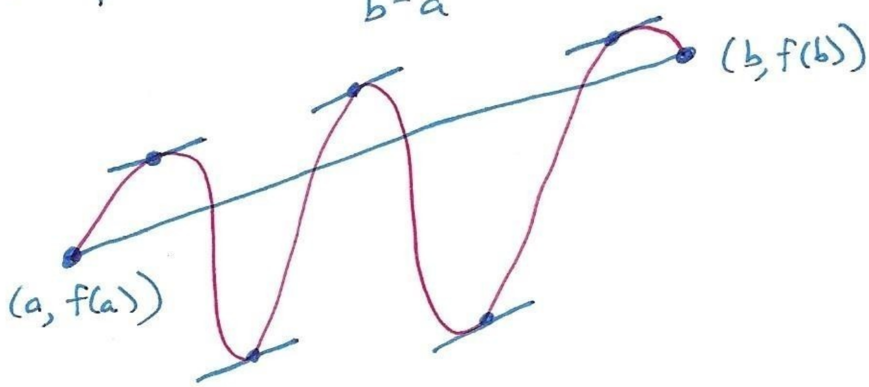
The Mean Value Theorem

If $f(x)$ is a function such that

① $f(x)$ is continuous on $a \le x \le b$, and

② $f(x)$ is differentiable on $a < x < b$

then there is at least one point $x = p$ in $a < x < b$ for which $f'(p) = \dfrac{f(b) - f(a)}{b - a}$.



eg An unknown function $f(x)$ has $f(1) = -2$ and $f'(x) \ge 3$ for all x. What is the minimum possible value of $f(6)$?

Consider the interval $1 \le x \le 6$. We are given that $f'(x)$ for all x, and therefore on $1 < x < 6$ we have $f(x)$ differentiable, while on $1 \le x \le 6$ we have $f(x)$ continuous by virtue of the result that differentiability implies continuity. Thus the Mean Value Theorem applies to $f(x)$.

Now we know that there exists a point $x=p$ where

$$f'(p) = \frac{f(6) - f(1)}{6-1}$$

$$= \frac{f(6) + 2}{5}$$

But since $f'(x) \geq 3$ for all x,

$$\frac{f(6)+2}{5} \geq 3$$

$$f(6) + 2 \geq 15 \rightarrow \boxed{f(6) \geq 13}$$